

Large deviations for random surfaces: the hyperbolic nature of Liouville Field Theory

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Abstract

Liouville Field Theory (LFT for short) is a two dimensional model of random surfaces, which is for instance involved in $2d$ string theory or in the description of the fluctuations of metrics in $2d$ Liouville quantum gravity. This is a probabilistic model that consists in weighting the classical shifted Free Field action with an interaction term involving a cosmological constant μ and what physicist call a background tachyon. This tachyon field is nothing but a Gaussian multiplicative chaos, formally the exponential of the Free Field times a constant γ , called the Liouville conformal factor. In this paper, we explain how to rigorously construct such a theory on the disk and review some of its intriguing properties, like the Knizhnik-Polyakov-Zamolodchikov formulae. The main input of our work is the study of the semiclassical limit of the theory. More precisely, when sending the Liouville conformal factor γ to 0 while keeping the quantity $\Lambda = \mu\gamma^2$ fixed, the so-called semiclassical limit regime, we derive exact formulas for the Laplace transform of the Liouville field. As a consequence, we prove that the Liouville field concentrates on the solution of the classical Liouville equation (involved in the uniformization theorem of surfaces) with prescribed negative (Ricci) scalar curvature $8\pi^2\Lambda$: we illustrate this statement by proving convergence in probability and by characterizing the leading fluctuations, which are Gaussian and massive. Though considered as an ansatz in the whole physics literature (see for instance [19, 41, 34, 53]), it seems that it is the first rigorous probabilistic derivation of the semiclassical limit of LFT. To complete this picture, we prove that this description of LFT as an hyperbolic geometry is rather sharp by establishing a large deviation principle with an explicit non trivial good rate function with a unique minimum located on the solution of the Liouville equation with curvature $8\pi^2\Lambda$.

Then we also we derive exact formulas for the Laplace transform of the Liouville field when we further weight the Liouville action with heavy matter operators. This procedure appears when computing the n -points correlation functions of LFT. This time, we show that the Liouville metric concentrates on metrics with prescribed negative curvature $8\pi^2\Lambda$ and conical singularities at the places of insertion. These metrics are obtained by solving the classical Liouville equation with additional sources. In this case, we also establish the convergence in probability, the characterization of fluctuations and a large deviation principle.

Key words or phrases: large deviation, Liouville equation, Gaussian multiplicative chaos, semiclassical limit, quantum field theory, Liouville field theory, singular Liouville equation.

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1 Introduction

It may be worth beginning this introduction with a brief overview of the considerable literature devoted to Laplace asymptotic expansions and large deviation principles for the canonical random paths: the Brownian motion in \mathbb{R}^d . To make things simple, the aim of these studies is to investigate the asymptotic behaviour as $\gamma \rightarrow 0$ of

$$\mathbb{E}[G(\gamma B)e^{-\gamma^{-2}F(\gamma B)}] \quad (1.1)$$

where B is a Brownian motion and F, G are general functionals. Schilder's pioneering work [50] (see also [42]) treated the full asymptotic expansion in the case of Wiener integrals. Similar results were obtained for conditioned Brownian paths (such as the Brownian bridge) by Davies and Truman [13, 14, 15, 16]. Ellis and Rosen [26, 27, 28] also developed further Laplace asymptotic expansions for Gaussian functional integrals. Then Azencott and Doss [3] used asymptotic expansions to study the semiclassical limit of the Schrödinger equation (see also Azencott [1, 2]).

These works initiated a long series (see for instance [7, 8, 9, 10]) and it is beyond the scope of this paper to review the whole literature until nowadays.

One of the aims of this paper is to initiate the study of Laplace asymptotics and large deviation principles in the realm of continuous random surfaces. There is an important conceptual difference between canonical random paths and canonical random surfaces. Whereas Brownian motion and its variants are rather nicely behaved, the canonical two dimensional random surface, i.e. the Gaussian Free Field (GFF), is much wilder: it cannot be defined pointwise for instance and must be understood as a random distribution. As a consequence, many nonlinear functionals defined solely on the space of continuous functions must be defined via renormalization techniques when applied to the GFF: see the book of Simon [51] for instance. In this paper, we consider probably the most natural framework of weighted random surfaces: the $2d$ -Liouville Field Theory (LFT). LFT is ruled by the (non critical, i.e. $\mu \neq 0$) Liouville action

$$S_L(\varphi) = \frac{1}{4\pi} \int_D [\langle \partial^{\hat{g}} \varphi, \partial^{\hat{g}} \varphi \rangle_{\hat{g}} + Q R_{\hat{g}} \varphi + 4\pi \mu e^{\gamma \varphi}] \lambda_{\hat{g}}(dx) \quad (1.2)$$

in the background metric \hat{g} ($\partial^{\hat{g}}$, $R_{\hat{g}}$ and $\lambda_{\hat{g}}$ stand for the gradient, curvature and volume form of the metric \hat{g}) with $Q = \frac{\gamma}{2} + \frac{2}{\gamma}$ and $\gamma \in]0, 2]$. This is a model describing random surfaces or metrics. Informally, the probability to observe a surface in $D\varphi$ is proportional to $e^{-S_L(\varphi)} D\varphi$ where $D\varphi$ stands for the "uniform measure" on surfaces. To give a rigorous meaning to this definition we have to interpret the term

$$\exp \left(- \frac{1}{4\pi} \int_D \langle \partial^{\hat{g}} \varphi, \partial^{\hat{g}} \varphi \rangle_{\hat{g}} \lambda_{\hat{g}}(dx) \right) D\varphi$$

as the law of the centered Free Field (see [22]). Therefore, you replace the Brownian motion in (1.1) by a shifted Free Field φ and the nonlinear functional F in (1.1) is the integrated exponential of this Free Field

$$\mathbb{E}[G(\gamma \varphi) e^{-\frac{4\pi\Lambda}{\gamma^2} \int_D e^{\gamma \varphi} \lambda_{\hat{g}}}], \quad (1.3)$$

Such an exponential, also called Liouville measure, is nothing but a Gaussian multiplicative chaos in $2d$. Recall that the theory of Gaussian multiplicative chaos, founded in 1985 by Kahane [35], enables to make sense of the exponential of the GFF though the exponential is not defined on the abstract functional space the GFF lives on. Our main motivation for considering this framework is to compute the semiclassical limit of $2d$ -Liouville Quantum Field Theory and establish a large deviation principle.

Before going into further details, we must mention that there is a sizeable literature on mathematical studies of discrete random surfaces in all dimensions: see for instance the recent review of Funaki [29]. In particular, within the (discrete) framework of gradient perturbations of the GFF, there has been an impressive series of results: large deviation principles (see [18]) or central limit theorems (see [40] for a convergence to the massless GFF in the whole space) for instance. Nonetheless, the results of this paper bear major differences with the discrete case and, in particular, cannot be derived from the discrete frameworks previously considered. Indeed, in the discrete case, one can work in nice functional spaces whereas, in the continuum setting of this paper, the GFF lives in the space of distributions and not in a functional space. Besides, an important aspect of our work is the derivation of an exact equivalent for the partition function (see (3.5) below) and more generally the Laplace transform of the field φ : this is a specific feature of the continuum setting which is essential in the problem of establishing exact relations for the three point correlation function, the celebrated DOZZ formula [21, 56] derived on the sphere (see [34] for a recent article on this problem).

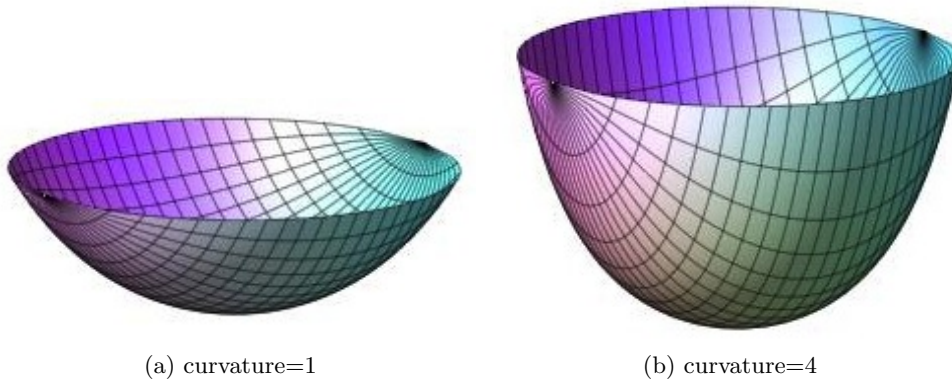


Figure 1: Two surfaces with negative curvature

Now recall that, in the physics literature, the Liouville action enables to describe random metrics in Liouville quantum gravity in the conformal gauge as introduced by Polyakov [43] (studied by David [12] and Distler-Kawai [19], see also the seminal work of Knizhnik-Polyakov-Zamolodchikov [37] in the light cone gauge) or in $2d$ string theory (see Klebanov's review [36] for instance). There are many excellent reviews on this topic [34, 41, 43, 53]. The rough idea is to couple the action of a conformal matter field (say a planar model of statistical physics at its critical point so as to become conformally invariant) to the action of gravity. This gives a couple of random variables $(e^{\gamma\varphi}\hat{g}, M)$, where the random metric $e^{\gamma\varphi}\hat{g}$ encodes the structure of the space and M stands for the matter field. Liouville quantum gravity in $2d$ can thus be seen as a toy model to understand in quantum gravity how the interaction with matter influences the geometry of space-time. In the conformal gauge and up to omitting some details, the law of this couple of random variables tensorizes [43, 12] and the marginal law of the metric $e^{\gamma\varphi}\hat{g}$ is given by the Liouville action (1.2). The only way the metric keeps track of its interaction with the matter field M is through the parameter γ , called Liouville conformal factor, which can be explicitly expressed in terms of the central charge c of the matter field using the celebrated KPZ result [37]

$$\gamma = \frac{\sqrt{25-c} - \sqrt{1-c}}{\sqrt{6}}. \quad (1.4)$$

Therefore, the influence of the matter is parameterized by γ . In a way, we will see that the geometry of space is encoded in the quantity $\mu\gamma^2$. More precisely, we study the so-called semiclassical limit, meaning the convergence of the field $\gamma\varphi$ when the parameter $\gamma \rightarrow 0$ while keeping fixed the quantity $\Lambda = \mu\gamma^2$ (thus $\mu \rightarrow \infty$). In the case where \hat{g} is flat or hyperbolic, we prove that the field $\gamma\varphi$ (resp. the measure $e^{\gamma\varphi} \lambda_{\hat{g}}(dx)$) converges in law towards the solution U (resp. $e^{U(x)} \lambda_{\hat{g}}(dx)$) to the so-called classical Liouville equation

$$\Delta_{\hat{g}} U - R_{\hat{g}} = 8\pi^2 \Lambda e^U. \quad (1.5)$$

Recall that equation (1.5) appears when looking for metrics in the conformal equivalence class of \hat{g} with prescribed Ricci scalar curvature $-8\pi^2 \Lambda$ and has for instance the following explicit form on the unit disk \mathbb{U} (equipped with the flat metric, i.e. take \hat{g} equal to the Euclidean metric)

$$U(x) = 2 \ln \frac{1-\alpha}{1-\alpha|x|^2}, \quad \text{with } \pi^2 \Lambda = \frac{\alpha}{(1-\alpha)^2},$$

with $\alpha \in]0, 1[$ when imposing Dirichlet boundary condition on $\partial\mathbb{U}$. The quantity $-8\pi^2\Lambda$ describes the expected curvature of the random metric $e^{\gamma\varphi}\hat{g}$ (see Figure 1) for small γ . The fact that this quantity is negative reflects the hyperbolic nature of the geometry of space.

We also characterize the leading order fluctuations around this hyperbolic geometry. They are Gaussian and massive in the sense that the rescaled field $\varphi - \gamma^{-1}U$ converges towards a massive free field in the metric $e^{U(x)}\hat{g}$. The mass of this free field is $8\pi^2\Lambda$ and thus exactly corresponds to minus the curvature: the more curved the space is, the more massive the Gaussian fluctuations are.

Then we investigate the possible deviations away from this hyperbolic geometry: we prove that the Liouville field satisfies a large deviation principle with an explicit good rate function, the Liouville action given by (1.2). This rate function is non trivial, admits a unique minimum on the solution to the Liouville equation with curvature $8\pi^2\Lambda$. The proof is based on computing the exact asymptotic expansion of expression (1.3) when G is the exponential of a linear function of φ : in fact, this exact expression (see (3.5) below when $G = 1$) is the main result of this paper as all the other results stem from this equivalent. In a way, this shows that this hyperbolic description of the geometry of the space is rather sharp as it shows that drifting away from the Liouville equation arises only with exponentially small probability.

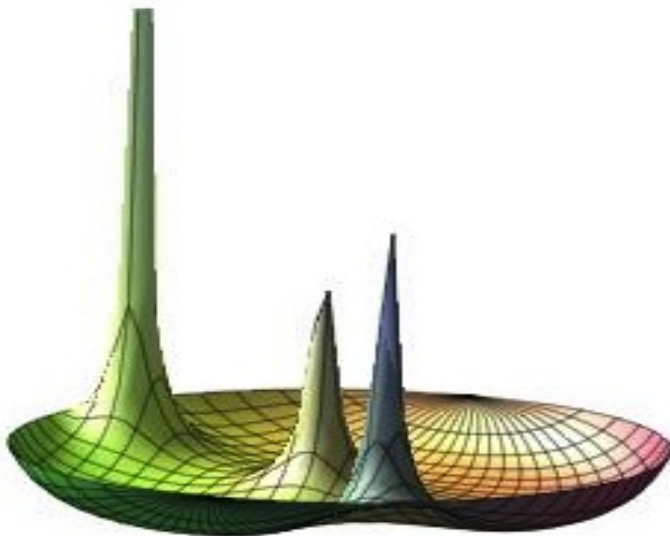


Figure 2: Surface with negative curvature and conical singularities

On the other hand, we investigate the Liouville action with heavy matter insertions. This means that we plug p exponential terms of the form $e^{\frac{\chi_i}{\gamma}X}$ (with $\chi_i \in [0, 2]$) in the Liouville action in order to compute the p -point correlation functions of LFT. We prove that the Liouville field $\gamma\varphi$ then concentrates on the solution of the Liouville equation with sources (where δ_{z_i} stands for the Dirac mass at z_i)

$$\Delta_{\hat{g}}U - R_{\hat{g}} = 8\pi^2\Lambda e^U - 2\pi \sum_{i=1}^p \chi_i \delta_{z_i} \quad U|_{\partial\mathbb{U}} = 0. \quad (1.6)$$

This equation appears when one looks for a metric with prescribed negative curvature $8\pi^2\Lambda$ and conical singularities at the points z_1, \dots, z_p (see Figure 2). Each source $\chi_i \delta_{z_i}$ creates a singularity

with shape $\sim \frac{1}{|x-z_i|^{\chi_i}}$ in the metric $e^{U(x)}\hat{g}$. Such singularities are called conical as they are locally isometric to a cone with "deficit angle" $\pi\chi_i$ (see Figure 3).

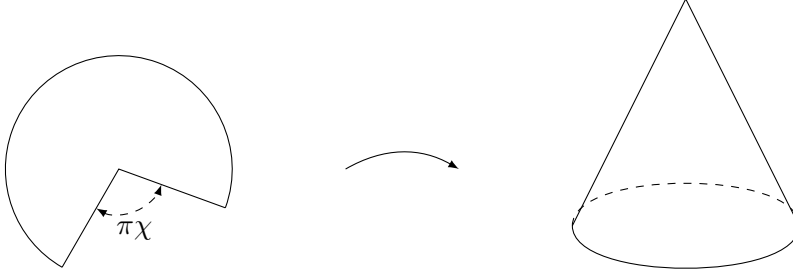


Figure 3: Cone with deficit angle $\pi\chi$. Glue isometrically the two boundary segments of the left-hand side figure to get the cone of the right-hand side figure. Such a cone is isometric to the complex plane equipped with the metric $ds^2 = |z|^{-\chi} dz d\bar{z}$.

Here again, the leading order fluctuations around this hyperbolic geometry with conical singularities are Gaussian and massive in the sense that the rescaled field $\varphi - \gamma^{-1}U$ converges towards a massive free field in the metric $e^{U(x)}\hat{g}$, where U is now the solution of (1.6). The mass of this free field is once again minus the curvature, namely $8\pi^2\Lambda$. We also establish a large deviation principle with an explicit good rate function, which is non trivial and admits a unique minimum on the solution of the Liouville equation with sources.

Conformal gravity in $4d$. Let us stress that an analog $4d$ -conformal field theory have been studied in the physics literature (see [33]) from quantized gravity. The dynamics are governed by the Wess-Zumino action and the Weyl action. Basically, the underlying idea is that the $4d$ Paneitz operator is conformally covariant and yields a notion of Q -curvature. To put it simply, we can consider the Euclidean background metric so that the Paneitz operator simply becomes the bilaplacian. The action then becomes

$$S_{WZW}(\varphi) = \frac{1}{16\pi^2} \int_D [\langle \Delta\varphi, \Delta\varphi \rangle + 16\pi^2 \mu e^{\gamma\varphi}] \lambda(dx), \quad (1.7)$$

which is the $4d$ analog of (1.2) in $4d$ in flat background metric. The important point for our purposes is that the corresponding free field action ($\mu = 0$) generates a log-correlated Gaussian field so that our approach applies word for word. In passing, we mention that such a theory shares fractal properties similar to $2d$ Liouville field theory, like the geometrical KPZ formula, as proved in [4, 46]. The semiclassical limit is described in terms of the equation

$$\Delta^2 = \Lambda e^U, \quad (1.8)$$

which is a prescription of constant (negative) Q -curvature. The reader may consult [20] for more on this topic of Q -uniformization of $4d$ -surfaces. Heavy matter operators may be added as well, leading to a perturbed equation (1.8) with additional sources (i.e. Dirac masses). This approach can also be generalized to even larger dimensions by considering the conformally covariant GJMS operators, which take the simple form $\Delta^{d/2}$ in even d -dimensional flat space.

Discussion on possible extensions or other geometries. Extra boundary terms

$$\frac{1}{2\pi} \oint_{\partial D} [Q r_{\hat{g}} \varphi + \kappa e^{\gamma\varphi/2}] d\ell,$$

where $d\ell$ stands for the length element on ∂D in the metric \hat{g} and $r_{\hat{g}}$ for the geodesic curvature on ∂D , may be considered as well in the Liouville action (1.2). These boundary terms rule the behaviour of the Liouville field on the boundary and gives rise to the boundary Liouville Field theory. We will refrain from considering these extra terms here: our purpose is to expose some aspects of LFT mainly to mathematicians and we wish to avoid these further complications. Yet, this would be a first natural (and non trivial) extension of our work.

In this paper, we focus on the unit disk with flat or hyperbolic geometry. Of particular interest is the construction of LFT on the sphere. It can be constructed by taking the large R limit of $2d$ Liouville quantum gravity on the disk with radius R and a curvature bump on its boundary (see [34]). In that case, the semi-classical limit exhibits some further interesting features. The point is that the limiting equations requires to construct a hyperbolic structure on the sphere, which is rather not inclined to support such a structure. This can be addressed by taking care of the nature of the insertions in the surface: there are some additional constraints on the insertions $(z_i, \chi_i)_i$, which are called Seiberg bound in the physics literature. This problem also receives a new growing interest in the community of differential geometry: the reader may consult for instance [5, 6, 54] and references therein for more on this topic and other closely related topics, like the Toda system. Indeed, another natural extension of our work could be to consider the large deviations of Toda field theories. In fact, tilting the free field measure with any nonlinear functional of the free field that yields interesting critical points for the Laplace method deserves to be investigated.

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2 Background and notations

2.1 Notations

Differential geometry: The standard gradient, Laplacian and Lebesgue measure on (a subdomain of) \mathbb{R}^2 are denoted by ∂ , Δ and $\lambda(dx)$ (and sometimes even dx). We will adopt the following notations related to Riemannian geometry throughout the paper. On a bounded domain D of \mathbb{R}^2 , a smooth function $\hat{g} : D \rightarrow]0, \infty[$ defines a scalar metric tensor by

$$(x, u, v) \in D \times \mathbb{R}^2 \times \mathbb{R}^2 \mapsto \hat{g}(x) \langle u, v \rangle,$$

where $\langle u, v \rangle$ stands for the canonical inner product on \mathbb{R}^2 . In what follows, we will denote by $\hat{g}(x)dx^2$ this metric tensor and sometimes, with a slight abuse of notation, identify $\hat{g}(x)dx^2$ with the function \hat{g} .

We can associate to this metric tensor a gradient $\partial^{\hat{g}}$, a Laplace-Beltrami operator $\Delta_{\hat{g}}$, a Ricci scalar curvature $R_{\hat{g}}$, and a volume form $\lambda_{\hat{g}}$, which are defined by:

$$\partial^{\hat{g}}\varphi(x) = \hat{g}(x)^{-1} \partial\varphi(x) \quad R_{\hat{g}}(x) = -\Delta_{\hat{g}} \ln \hat{g}(x) \quad (2.1)$$

$$\Delta_{\hat{g}}\varphi(x) = \hat{g}(x)^{-1} \Delta\varphi(x) \quad \int_D \varphi(x) \lambda_{\hat{g}}(dx) = \int_D \varphi(x) \hat{g}(x) \lambda(dx). \quad (2.2)$$

We denote by $\langle \partial^{\hat{g}}\varphi, \partial^{\hat{g}}\psi \rangle_{\hat{g}}$ the pairing of two gradients $\partial^{\hat{g}}\varphi, \partial^{\hat{g}}\psi$ in the metric \hat{g} , that is

$$\langle \partial^{\hat{g}}\varphi, \partial^{\hat{g}}\psi \rangle_{\hat{g}}(x) = \hat{g}(x)^{-1} \langle \partial\varphi(x), \partial\psi(x) \rangle. \quad (2.3)$$

Green function and conformal maps: The Green function on a domain D will be denoted by $G_D(x, y)$. By definition, the Green function is the unique function which solves the following equation for all $x \in D$:

$$\Delta_y G_D(x, y) = -2\pi\delta_x, \quad G_D(x, \cdot) = 0 \text{ on } \partial D.$$

Note that with this convention $G_D(x, y) = \ln \frac{1}{|y-x|} + \varphi(x, y)$ where φ is a smooth function on D (not smooth on the whole boundary ∂D). By conformal map $\psi : \tilde{D} \rightarrow D$, we will always mean a bijective bi-holomorphic map from \tilde{D} onto D . Recall that the Green function is conformally invariant in the sense that $G_D \circ \psi = G_{\tilde{D}}$.

Functional spaces: $C_c^\infty(D)$ stands for the space of smooth compactly supported functions on D . We denote $\mathbb{L}^p(D)$ the standard space of functions u such that $|u|^p$ is integrable. Classically, if D is a (say) smooth bounded domain, we define the space $H_0^1(D)$ as the completion of $C_c^\infty(D)$ with respect to the (squared) norm $|\varphi|_{H^1}^2 = \int_D |\partial\varphi|^2 dx$. Let us recall a few facts on $H_0^1(D)$ and its dual $H^{-1}(D)$ which we need in the paper: see [22, section 4.2] for instance. The space $H^{-1}(D)$ is defined as the Banach space of continuous linear functionals f on $H_0^1(D)$ equipped with the norm

$$|f|_{H^{-1}} = \sup_{\varphi \in H_0^1(D), |\varphi|_{H^1} \leq 1} f(\varphi)$$

where we denote $f(\varphi)$ the distribution f applied at φ . The dual space of $(H^{-1}(D), |\cdot|_{H^{-1}})$ is once again a Banach space, which is isometric to $H_0^1(D)$ and we will make the standard abuse of notations to identify this Banach space with $H_0^1(D)$. The space $H_0^1(D)$ can then be equipped with the weak* topology, i.e. the topology induced by the linear functionals $\varphi \in H_0^1(D) \mapsto f(\varphi)$ for all f in $H^{-1}(D)$: see [17, Appendix B]. This topology coincides with the standard weak topology on $H_0^1(D)$. We will use this remark when establishing the large deviation principle.

2.2 Free Fields

Let \mathbb{P}, \mathbb{E} denote the probability law and expectation of a standard probability space; the corresponding space of variables Z such that $|Z|^p$ is integrable will be denoted by \mathbb{L}_p . On this space, the centered Gaussian Free field X with mass $m \geq 0$ on a planar domain $D \subset \mathbb{R}^2$ and Dirichlet boundary condition is the Gaussian field whose covariance function is given by the Green function G_D^m (recall that, for $m \equiv 0$, we denote $G_D = G_D^0$) of the problem

$$\Delta u - mu = -2\pi f \text{ on } D, \quad u|_{\partial D} = 0,$$

where $m \geq 0$ is a function defined on D . When the mass satisfies $m \neq 0$, one usually talks about Massive Free Field (MFF for short) whereas one rather uses the terminology Gaussian Free Field (GFF for short) for the massless field with $m \equiv 0$. Therefore, for any smooth compactly supported functions f, h on D

$$\mathbb{E} [X(f)X(h)] = \iint_{D \times D} f(x) G_D^m(x, y) h(y) dx dy.$$

Almost surely, the GFF lives on the space $H^{-1}(D)$ (see [22]).

Remark 2.1. In fact, X belongs to the standard Sobolev space $H^{-s}(D)$ for all $s > 0$ (see [22] for further details) but for simplicity, we refrain from considering this framework. Many theorems of this paper could in fact be strengthened to the topology of $H^{-s}(D)$; for instance, this is the case for the large deviation result, i.e. Theorem 3.4, by using [17, Theorem 4.2.4] which enables to strengthen topologies in large deviation principles.

Remark 2.2. We could treat other boundary conditions as well but for simplicity, we restrict to the case of Dirichlet boundary conditions. In the case when the action possesses boundary terms, it is more relevant to consider a GFF with Neumann boundary conditions in the following.

In what follows, we need to consider cut-off approximations of the Gaussian free field X on D . The cut-off may be any of the following:

-White Noise (WN) (see [46, 38, 45]): The Green function G_D on D can be written as

$$G_D(x, y) = \pi \int_0^\infty p(r, x, y) dr.$$

where $p(t, x, y)$ will denote the transition densities of the Brownian motion on D killed upon touching ∂D . A formal way to define the Gaussian field X is to consider a white noise W on $\mathbb{R}_+ \times D$ and define

$$X(x) = \sqrt{\pi} \int_0^\infty \int_D p(r/2, x, y) W^X(dr, dy). \quad (2.4)$$

We define the approximations X_ε by integrating over $(\varepsilon^2, \infty) \times D$ in (2.4) instead of $(0, \infty) \times D$. The covariance function for these approximations is given by

$$\mathbb{E}[X_\varepsilon(x)X_{\varepsilon'}(y)] = \pi \int_{\varepsilon^2 \vee \varepsilon'^2}^\infty p(r, x, y) dr. \quad (2.5)$$

-Circle Average (CA) (see [23]): We introduce the circle averages $(X_\varepsilon)_{\varepsilon \in [0,1]}$ of radius ε , i.e. $X_\varepsilon(x)$ stands for the mean value of X on the circle centered at x with radius ε . We could also consider more general mollifiers (see [48, 49]).

-Orthonormal Basis Expansion (OBE) (see [22, 23, 38, 45]): We consider an orthonormal basis $(f_k)_{k \geq 1}$ of $H_0^1(D)$ made up of continuous functions and the projections of X onto this orthonormal basis, namely we define the sequence of i.i.d. Gaussian random variables:

$$\varepsilon_k = \frac{1}{2\pi} \int_D \langle \partial X(x), \partial f_k(x) \rangle dx.$$

The projections of X onto the span of $\{f_1, \dots, f_n\}$ are given by $X_n(x) = \sum_{k=1}^n \varepsilon_k f_k(x)$.

In any of the above three cases, the family of cut-off approximations will be denoted by $(X_\varepsilon)_\varepsilon$ (with $\varepsilon = e^{-n}$ in the case of (OBE)).

2.3 Gaussian multiplicative chaos

For the three possible cut-off approximations $(X_\varepsilon)_\varepsilon$ of the GFF and for $\gamma \in [0, 2]$, we consider the random measure on D defined by

$$e^{\gamma X(x)} dx = \lim_{\varepsilon \rightarrow 0} \varepsilon^{\frac{\gamma^2}{2}} e^{\gamma X_\varepsilon(x)} dx. \quad (2.6)$$

The limit holds almost surely and is understood in the sense of weak convergence of measures. This has been proved in [35] for the cut-off family (WN) and (OBE) and in [23] for (CA). The limit is non trivial if and only if $\gamma < 2$ (see [35]). For these three possible cutoffs, the limiting objects $(X, e^{\gamma X(x)} dx)$ that you get by taking the limit as $\varepsilon \rightarrow 0$ have the same law [45].

The Wick Notation

In the paper we make extensive use of Wick notation for the exponential. If Z is a Gaussian variable with mean zero and variance σ , its Wick n -th power ($n \in \mathbb{N}$) is defined by

$$: Z^n : = \sum_{m=0}^{\lfloor n/2 \rfloor} \frac{(-1)^m n!}{m!(n-2m!)2^m} \sigma^{2m} Z^{n-2m} = \sigma^n H_n(\sigma^{-1} Z) \quad (2.7)$$

where H_n is the n -th Hermit Polynomial. If Z is not centered then $: Z^n :$ is understood as $: Z^n : = : \tilde{Z}^n :$ where $\tilde{Z} := Z - \mathbb{E}[Z]$.

This definition is designed to make the Wick monomials orthogonal to each other. More precisely if (Z, Y) is a Gaussian vector we have

$$\mathbb{E}[: Z^n :: Y^m :] = n! \mathbf{1}_{n=m} \mathbb{E}[ZY]^n. \quad (2.8)$$

The Wick exponential is defined formally as the result of the following expansion in Wick's power

$$: e^{\gamma Z} : = \sum_{n=0}^{\infty} \frac{\gamma^n : Z^n :}{n!}. \quad (2.9)$$

A bit of combinatorics with Wicks monomial leads to the following identity

$$: e^{\gamma Z} := \exp\left(\gamma \tilde{Z} - \frac{\sigma^2 \gamma^2}{2}\right). \quad (2.10)$$

Most of the time we will use the Wick notation for Gaussian fields that are distributions rather than Gaussian variables, but we specify the meaning of this notation below.

Wick Notation for Gaussian Fields

We consider a Free Field X defined on a planar domain D (we stress that the basics below extend without changes to any other log-correlated Gaussian field). We define the Wick powers and the Wick exponential as a distribution on D , by taking the limit of cut-off approximations of X constructed in Section 2.2. Indeed from the formula (2.8), the reader can check that for any smooth function u , and any n the sequence

$$\int_D : X_\varepsilon^n(x) : u(x) dx \quad (2.11)$$

is Cauchy in \mathbb{L}_2 , and thus admits a limit understood as a random distribution $: X^n :$ acting on u . For $\gamma < 2$, one can also consider the limit $: e^{\gamma X(x)} : dx$ in the sense of weak convergence of measures of the family $(: e^{\gamma X_\varepsilon(x)} : dx)_\varepsilon$ and one can check that

$$e^{\gamma X(x)} dx = : e^{\gamma X} : C(x, \mathbb{U})^{\gamma^2/2} dx, \quad (2.12)$$

where $C(x, \mathbb{U})$ denotes the conformal radius and the measure is defined in subsection 2.3. Notice that for $\gamma < \sqrt{2}$, the limit can also be obtained from the series expansion (2.9): for all $u \in \mathbb{L}^p(D)$ for some $p > 1$

$$\int_D : e^{\gamma X(x)} : u(x) dx = \sum_{n \geq 0} \frac{\gamma^n}{n!} \int_D : X^n(x) : u(x) dx, \quad (2.13)$$

where the above sum converges in \mathbb{L}_2 .

3 Semiclassical limit

3.1 The semiclassical limit

In this section and for pedagogical purpose, we make one simplification by not taking into account a possibly curved space. Yet, this is not a big restriction as it plays no part in what follows. Furthermore, a complete framework is described in section 5.

We equip the unit disk \mathbb{U} with the flat metric, i.e. the metric associated to the metric tensor $g = 1$ on \mathbb{U} . We consider a GFF X on \mathbb{U} with Dirichlet boundary condition. We consider a cosmological constant $\mu \geq 0$ and a Liouville conformal factor $\gamma \in]0, 2]$. We set

$$Q = \frac{2}{\gamma} + \frac{\gamma}{2}.$$

We define the law $\mathbb{P}_{\mu, \gamma}$ of the Liouville field X on \mathbb{U} associated to (μ, γ) as the tilted version of \mathbb{P} as follows:

$$\mathbb{E}_{\mu, \gamma}[F(X)] = Z_{\mu, \gamma}^{-1} \mathbb{E}\left[F(X) \exp\left(-4\pi\mu \int_{\mathbb{U}} e^{\gamma X(x)} dx\right)\right] \quad (3.1)$$

where

$$Z_{\mu, \gamma} = \mathbb{E}\left[\exp\left(-4\pi\mu \int_{\mathbb{U}} e^{\gamma X(x)} dx\right)\right]$$

and F is any bounded continuous functional on $H^{-1}(\mathbb{U})$.

Our aim is to determine the asymptotic behavior of the fields γX when γ tends to zero and μ tends to infinity simultaneously while satisfying the relation

$$\mu\gamma^2 = \Lambda, \quad (3.2)$$

for a fixed positive Λ .

We claim

Theorem 3.1. *Assume that $\gamma \rightarrow 0, \mu \rightarrow \infty$ under the constraint (3.2). The field γX concentrates on the solution of the classical Liouville equation*

$$\Delta U = 8\pi^2 \Lambda e^U \quad (3.3)$$

with zero boundary condition on \mathbb{U} . More precisely

1. *The partition function has the following asymptotic behavior at the exponential scale*

$$\lim_{\gamma \rightarrow 0} \gamma^2 \ln Z_{\mu, \gamma} = -\frac{1}{4\pi} \int_{\mathbb{U}} (|\partial U(x)|^2 + 16\pi^2 \Lambda e^{U(x)}) dx = F(\Lambda). \quad (3.4)$$

2. *More precisely we have the following equivalent as $\gamma \rightarrow 0$*

$$Z_{\mu, \gamma} \sim e^{\gamma^{-2} F(\Lambda)} \exp\left(-2\pi\Lambda \int_{\mathbb{U}} e^{U(x)} \ln C(x, \mathbb{U}) dx\right) \mathbb{E}\left[\exp\left(-2\pi\Lambda \int_{\mathbb{U}} e^{U(x)} : X(x)^2 : dx\right)\right], \quad (3.5)$$

where $: X^2 :$ is the standard Wick-ordered square field, i.e. $: X(x)^2 := \lim_{\varepsilon \rightarrow 0} X_\varepsilon(x)^2 - \mathbb{E}[X_\varepsilon(x)^2]$ where X_ε is the cut-off field (see subsection 2.2).

3. *The field γX converges in probability in $H^{-1}(\mathbb{U})$ as $\gamma \rightarrow 0$ towards U .*

4. Both random measures : $e^{\gamma X} : dx$ and $e^{\gamma X} dx$ converge in law in the sense of weak convergence of measures towards $e^{U(x)} dx$ as $\gamma \rightarrow 0$.
5. the field $X - \gamma^{-1}U$ converges in law in $H^{-1}(\mathbb{U})$ as $\gamma \rightarrow 0$ towards a Massive Free Field in the metric $\hat{g} = e^{U(x)} dx^2$ with Dirichlet boundary condition and mass $8\pi^2\Lambda$, that is a Gaussian field with covariance kernel given by the Green function of the operator $2\pi(8\pi^2\Lambda - \Delta_{\hat{g}})^{-1}$ with Dirichlet boundary conditions.

Remark 3.2. The above theorem shows in a way that the metric $e^{\gamma X(x)} dx^2$ converges as $\gamma \rightarrow 0$ towards the metric on the disk with negative curvature $-8\pi^2\Lambda$. Actually, we only treat here the case of the curvature or volume form of the metric. But the same argument can be adapted for instance to prove the convergence of the associated Brownian motion defined in [30, 31].

3.2 The large deviation principle

Now we focus on a Large Deviation Principle. Recall that U is the solution to the classical Liouville equation (3.3). For $f \in H_0^1(\mathbb{U})$, we consider the weak solution V of the perturbed Liouville equation (see Theorem A.1)

$$\Delta V = 8\pi^2\Lambda e^{V(x)} - 2\pi f(x), \quad \text{with } V|_{\partial\mathbb{U}} = 0, \quad (3.6)$$

and we set

$$F(\Lambda, f) = -\frac{1}{4\pi} \int_{\mathbb{U}} (|\partial V(x)|^2 + 16\pi^2\Lambda e^{V(x)}) + \int_{\mathbb{U}} f(x)(V(x) - U(x)) dx.$$

In the course of the proof of our large deviation result: Theorem 3.4, we will check that the mapping $f \in H_0^1(\mathbb{U}) \mapsto F(\Lambda, f) - F(\Lambda)$ is convex, Gâteaux-differentiable and weakly lower semi-continuous (for the weak* topology).

We define its Fenchel-Legendre transform as follows by

$$\forall h \in H^{-1}(\mathbb{U}), \quad I^*(h) = \sup_{f \in H_0^1(\mathbb{U})} \{h(f) - F(\Lambda, f) + F(\Lambda)\}. \quad (3.7)$$

Proposition 3.3. The function I^* is a good rate function with explicit expression

$$I^*(h) = \begin{cases} E(U + h) - E(U), & \text{if } h \in H_0^1(\mathbb{U}), \\ +\infty, & \text{otherwise,} \end{cases}$$

where

$$\forall u \in H_0^1(\mathbb{U}), \quad E(u) = \frac{1}{4\pi} \int_{\mathbb{U}} (|\partial u(x)|^2 + 16\pi^2\Lambda e^{u(x)}) dx.$$

In particular, we have $I^*(h) > 0$ except for $h = 0$.

The fact that I^* vanishes only for $h = 0$ is important because this entails that the forthcoming LDP provides non trivial bounds as soon as the set A has non empty interior and $0 \notin \bar{A}$.

Theorem 3.4. Assume that $\gamma \rightarrow 0, \mu \rightarrow \infty$ under the constraint (3.2). Set $Y_\gamma = \gamma X - U$. The following LDP holds with good rate function I^* on the space $H^{-1}(\mathbb{U})$ equipped with the norm $|\cdot|_{H^{-1}}$

$$-\inf_{h \in \bar{A}} I^*(h) \leq \gamma^2 \liminf_{\gamma \rightarrow 0} \mathbb{P}_{\mu, \gamma}(Y_\gamma \in A) \leq \gamma^2 \limsup_{\gamma \rightarrow 0} \mathbb{P}_{\mu, \gamma}(Y_\gamma \in A) \leq -\inf_{h \in \bar{A}} I^*(h)$$

for each Borel subset A of $H^{-1}(\mathbb{U})$.

3.3 Proofs

Proof of Theorem 3.1. We first compute the limit of the partition function $Z_{\mu,\gamma}$.

$$\mathbb{E}\left[\exp\left(-\frac{4\pi\Lambda}{\gamma^2}\int_{\mathbb{U}}e^{\gamma X(x)}dx\right)\right]=\mathbb{E}\left[\exp\left(-\frac{4\pi\Lambda}{\gamma^2}\int_{\mathbb{U}}:e^{\gamma X(x)}:C(x,\mathbb{U})^{\frac{\gamma^2}{2}}dx\right)\right].$$

We define $Y = Y_\gamma$ as follows

$$Y(x) = X - \frac{U}{\gamma} \quad (3.8)$$

Where U is the solution of (3.3). Note that this implies in particular that

$$U(x) = -4\pi\Lambda\int_{\mathbb{U}}e^{U(y)}G_{\mathbb{U}}(x,y)dy. \quad (3.9)$$

We have

$$\begin{aligned} & \mathbb{E}\left[\exp\left(-\frac{4\pi\Lambda}{\gamma^2}\int_{\mathbb{U}}e^{\gamma X(x)}dx\right)\right] \\ &= \mathbb{E}\left[e^{-\frac{4\pi\Lambda}{\gamma^2}\int_{\mathbb{U}}e^{U(x)}(1+\gamma Y(x))dx}e^{-\frac{4\pi\Lambda}{\gamma^2}\int_{\mathbb{U}}e^{\gamma X(x)}-e^{U(x)}(1+\gamma Y(x))dx}\right] \\ &= e^{-\frac{4\pi\Lambda}{\gamma^2}\int_{\mathbb{U}}e^{U(x)}(1-U(x))dx+\frac{8\pi^2\Lambda^2}{\gamma^2}\int_{\mathbb{U}^2}e^{U(x)+U(y)}G_{\mathbb{U}}(x,y)dx dy} \\ & \times \mathbb{E}\left[e^{-\frac{4\pi\Lambda}{\gamma}\int_{\mathbb{U}}e^{U(x)}X(x)dx-\frac{8\pi^2\Lambda^2}{\gamma^2}\int_{\mathbb{U}^2}e^{U(x)+U(y)}G_{\mathbb{U}}(x,y)dx dy}e^{-\frac{4\pi\Lambda}{\gamma^2}\int_{\mathbb{U}}e^{\gamma X(x)}-e^{U(x)}(1+\gamma Y(x))dx}\right]. \end{aligned} \quad (3.10)$$

The first exponential term in the expectation

$$e^{-\frac{4\pi\Lambda}{\gamma}\int_{\mathbb{U}}e^{U(x)}X(x)dx-\frac{8\pi^2\Lambda^2}{\gamma^2}\int_{\mathbb{U}^2}e^{U(x)+U(y)}G_{\mathbb{U}}(x,y)dx dy}$$

is a Girsanov transform term. It has the effect of shifting the field X by a function which is equal to

$$-\frac{4\pi\Lambda}{\gamma}\int_{\mathbb{U}}e^{U(x)}G_{\mathbb{U}}(x,y)dx = \frac{U(x)}{\lambda}. \quad (3.11)$$

Hence after this shift, Y becomes a centered field, and the expectation in the last line of (3.10) is equal to

$$\mathbb{E}\left[\exp\left(-\frac{4\pi\Lambda}{\gamma^2}\int_{\mathbb{U}}e^{U(x)}(e^{\gamma X(x)}-1-\gamma X(x))dx\right)\right].$$

For the term in front of the expectation, from (3.9) we have the following simplification

$$\begin{aligned} 8\pi^2\Lambda^2\int_{\mathbb{U}^2}e^{U(x)+U(y)}G_{\mathbb{U}}(x,y)dx dy &= -2\pi\Lambda\int_{\mathbb{U}}e^{U(x)}U(x)dx = -\frac{1}{4\pi}\int_{\mathbb{U}}U(x)\Delta U(x)dx \\ &= \frac{1}{4\pi}\int_{\mathbb{U}}|\partial U(x)|^2dx, \end{aligned}$$

and thus it is equal to

$$\exp\left(-\frac{1}{4\pi\gamma^2}\int_{\mathbb{U}}(|\partial U(x)|^2+16\pi^2\Lambda e^{U(x)})dx\right). \quad (3.12)$$

The computation of the partition function (item 1. and item 2.) is completed with the following lemma, the proof of which is postponed after the end of the proof of Theorem 3.1:

Lemma 3.5. *For any bounded positive function g on \mathbb{U} one has*

$$\begin{aligned} \lim_{\gamma \rightarrow 0} \mathbb{E} \left[\exp \left(-\frac{1}{\gamma^2} \int_{\mathbb{U}} g(x) (e^{\gamma X(x)} - 1 - \gamma X(x)) dx \right) \right] \\ = e^{-\int_{\mathbb{U}} \ln C(x, \mathbb{U}) dx} \mathbb{E} \left[\exp \left(-\Lambda \int_{\mathbb{U}} g(x) : X(x)^2 : dx \right) \right], \end{aligned} \quad (3.13)$$

where $: X^2 :$ is the standard Wick-ordered square field defined in Section 2.3.

Remark 3.6. *Before going back to the proof of Theorem 3.1, let us make a few comments. First, observe that the expression in the exponential is not positive as the elementary inequality $e^u - 1 - u \geq 0$ might suggest. Indeed one should not forget that here $e^{\gamma X} dx$ is defined via a renormalization procedure (recall (2.6)). This being clear, let us shortly explain why (3.13) holds. We have*

$$\begin{aligned} \exp \left(-\frac{1}{\gamma^2} \int_{\mathbb{U}} g(x) (e^{\gamma X(x)} - 1 - \gamma X(x)) dx \right) \\ = \exp \left(\frac{1}{\gamma^2} \int_{\mathbb{U}} g(x) : e^{\gamma X(x)} : (1 - (C(x, \mathbb{U}))^{\gamma^2/2}) dx \right) \\ \times \exp \left(-\frac{1}{\gamma^2} \int_{\mathbb{U}} g(x) (: e^{\gamma X(x)} : - 1 - \gamma X(x)) dx \right). \end{aligned} \quad (3.14)$$

When γ tends to zero $(1 - (C(x, \mathbb{U}))^{\gamma^2/2})$ is equivalent to $-\gamma^2/2 \ln C(x, \mathbb{U})$. According to the expansion (2.13), it also makes sense to say that $: e^{\gamma X(x)} : dx \sim dx$ in some sense as γ goes to 0 so that the first term should converge to $e^{-\frac{1}{2} \int_{\mathbb{U}} g(x) \ln C(x, \mathbb{U}) dx}$.

As for the second term (2.13) tells us that

$$\gamma^{-2} g(x) (: e^{\gamma X(x)} : - 1 - \gamma X(x)) dx \sim \frac{g(x)}{2} : X^2(x) : dx \quad (3.15)$$

which indicates convergence.

The difficult part of the job is then to show that the formal equivalent above are rigorous in a sense, and also that lower order terms do not change the behavior of the Laplace transform on the left-hand side of (3.13).

Let us consider a continuous bounded function F on the space $H^{-1}(\mathbb{U})$. The same computation shows that

$$\begin{aligned} \mathbb{E} \left[F(\gamma X) \exp \left(-\frac{4\pi\Lambda}{\gamma^2} \int_{\mathbb{U}} e^{\gamma X(x)} dx \right) \right] = \exp \left(-\frac{1}{4\pi\gamma^2} \int_{\mathbb{U}} (|\partial U(x)|^2 + 16\pi^2 \Lambda e^{U(x)}) dx \right) \\ \times \mathbb{E} \left[F(\gamma X + U) \exp \left(-\frac{4\pi\Lambda}{\gamma^2} \int_{\mathbb{U}} e^{U(x)} (e^{\gamma X(x)} - 1 - \gamma X(x)) dx \right) \right]. \end{aligned} \quad (3.16)$$

As γX converges in probability towards 0, it is plain to deduce from Lemma 3.5 again that the expectation in the above right-hand side behaves when $\gamma \rightarrow 0$ as follows

$$\begin{aligned} \lim_{\gamma \rightarrow 0} \mathbb{E} \left[F(\gamma X + U) \exp \left(-\frac{4\pi\Lambda}{\gamma^2} \int_{\mathbb{U}} e^{U(x)} (e^{\gamma X(x)} - 1 - \gamma X(x)) dx \right) \right] \\ = F(U) e^{-2\pi\Lambda \int_{\mathbb{U}} e^{U(x)} \ln C(x, \mathbb{U}) dx} \mathbb{E} \left[\exp \left(-2\pi\Lambda \int_{\mathbb{U}} e^{U(x)} : X(x)^2 : dx \right) \right]. \end{aligned} \quad (3.17)$$

This shows the convergence in law of the field γX towards U . Hence item 3. Item 4 can be proved in the same way.

Now we focus on item 5. We use again the notation

$$Y = X - \gamma^{-1}U.$$

From (3.16), we have

$$\lim_{\gamma \rightarrow 0} \mathbb{E}_{\mu, \gamma}[F(Y)] = Z^{-1} \mathbb{E}[F(X) \exp(-2\pi\Lambda \int_{\mathbb{U}} e^{U(x)} : X(x)^2 : dx)] \quad (3.18)$$

where

$$Z := \mathbb{E}[\exp(-2\pi\Lambda \int_{\mathbb{U}} e^{U(x)} : X(x)^2 : dx)]. \quad (3.19)$$

Now we claim:

Lemma 3.7. *Under the tilted probability measure*

$$\tilde{\mathbb{P}} = Z_{\alpha}^{-1} e^{-\alpha \int_{\mathbb{U}} e^{U(x)} : X(x)^2 : dx} \mathbb{P}, \quad Z_{\alpha} = \mathbb{E}[e^{-\alpha \int_{\mathbb{U}} e^{U(x)} : X(x)^2 : dx}]$$

the field X has the law of a Massive Free Field in the metric $g = e^{U(x)} dx^2$ with Dirichlet boundary condition and mass $4\pi\alpha$, that is a Gaussian field with covariance kernel given by the Green function of the operator $2\pi(4\pi\alpha - \Delta_g)^{-1}$.

The proof of Theorem 3.1 is over. □

Proof of Lemma 3.7. Let \hat{g} be the metric tensor $e^{U(x)} dx^2$. Let $(\lambda_j)_j$ be the non-decreasing sequence of eigenvalues of $-(2\pi)^{-1}\Delta_{\hat{g}}$ with Dirichlet boundary conditions (with repetition if necessary to take into account multiple eigenvalue). Let $(e_j)_j$ be an orthogonal sequence of eigenvectors associated to λ_j normalized to 1 in the $\mathbb{L}^2(\mathbb{U}, \lambda_{\hat{g}})$ sense, i.e. $\int_{\mathbb{U}} e_j(x)^2 \lambda_{\hat{g}}(dx) = 1$. Note that the sequence $(e_j)_j$ is orthogonal in $\mathbb{L}^2(\mathbb{U}, \lambda_{\hat{g}})$ and in the Sobolev space $H_0^1(\mathbb{U})$ (see [11, chapter 7]). Recall that $\lambda_j \sim Cj$ as j goes to infinity according to the Weyl asymptotic formula. Then we have

$$X(x) = \sum_{j=1}^{\infty} \frac{e_j(x)}{\sqrt{\lambda_j}} \varepsilon_j$$

where $(\varepsilon_j)_j$ is an i.i.d. sequence of standard Gaussian variables given by $\varepsilon_j = \langle X, e_j \rangle_{H^1}$. In this case, we have

$$\int_{\mathbb{U}} : X(x)^2 : e^{U(x)} dx = \int_{\mathbb{U}} \left(\left(\sum_{j=1}^{\infty} \frac{e_j(x)}{\sqrt{\lambda_j}} \varepsilon_j \right)^2 - \sum_{j=1}^{\infty} \frac{e_j(x)^2}{\lambda_j} \right) e^{U(x)} dx = \sum_{j=1}^{\infty} \frac{\varepsilon_j^2 - 1}{\lambda_j}.$$

Therefore we have

$$Z_{\alpha} = \mathbb{E}[e^{-\alpha \int_{\mathbb{U}} : X(x)^2 : e^{U(x)} dx}] = \prod_j \mathbb{E}[e^{-\frac{\alpha}{\lambda_j} \varepsilon_j^2}] e^{\frac{\alpha}{\lambda_j}} = \prod_j \sqrt{\frac{\lambda_j}{\lambda_j + 2\alpha}} e^{\frac{\alpha}{\lambda_j}},$$

which converges if $\alpha > -\frac{\lambda_1}{2}$. By computing the Laplace transform of the field X under $\tilde{\mathbb{P}}$, it is plain to see that the field is Gaussian. It remains to identify the covariance structure

$$\begin{aligned}\tilde{\mathbb{E}}[X(x)X(y)] &= \mathbb{E}[X(x)X(y)e^{-\alpha \int_D :X(x)^2: e^{U(x)} dx}] / Z_\alpha \\ &= \sum_{j=1}^{\infty} \frac{e_j(x)e_j(y)}{\lambda_j} \frac{\mathbb{E}[\varepsilon_j^2 e^{-\alpha \sum_{k=1}^{\infty} \frac{\varepsilon_k^2 - 1}{\lambda_k}}]}{Z_\alpha} \\ &= \sum_{j=1}^{\infty} \frac{e_j(x)e_j(y)}{\lambda_j} \sqrt{\frac{\lambda_j + 2\alpha}{\lambda_j}} e^{-\frac{\alpha}{\lambda_j}} \mathbb{E}[\varepsilon_j^2 e^{-\alpha \frac{\varepsilon_j^2 - 1}{\lambda_j}}] \\ &= \sum_{j=1}^{\infty} \frac{e_j(x)e_j(y)}{\lambda_j + 2\alpha}.\end{aligned}$$

Hence the law of X is that of the Massive Free Field in the metric $e^{U(x)} dx^2$ conditioned to be 0 on the boundary of \mathbb{U} with mass $2\alpha \times 2\pi$. \square

Proof of Lemma 3.5.

The first step is to prove that the random variable

$$H_\gamma = \frac{1}{\gamma^2} \int_{\mathbb{U}} (e^{\gamma X(x)} - 1 - \gamma X(x) - \frac{\gamma^2}{2} \ln C(x, \mathbb{U}) - \frac{\gamma^2}{2} : X(x)^2 :) g(x) dx$$

converges in \mathbb{L}_2 towards 0 as $\gamma \rightarrow 0$. We have

$$\begin{aligned}H_\gamma &= \frac{1}{\gamma^2} \int_{\mathbb{U}} : e^{\gamma X(x)} : \left(C(x, \mathbb{U})^{\gamma^2/2} - 1 \right) - \frac{\gamma^2}{2} \ln C(x, \mathbb{U}) dx \\ &\quad \frac{1}{\gamma^2} \int_{\mathbb{U}} (: e^{\gamma X(x)} : - 1 - \gamma X(x) - \frac{\gamma^2}{2} : X(x)^2 :) g(x) dx\end{aligned}$$

It is rather straightforward to check the convergence to zero in \mathbb{L}_2 of the first term. As for the term of the second line, by the expansion (2.13) and the orthogonality of Wick polynomials (2.8), its variance is equal to

$$\sum_{n \geq 3} \frac{1}{n!} \gamma^{2n-4} \int_{\mathbb{U}^2} (G_{\mathbb{U}}^n(x, y))^n g(x) g(y) dx dy, \quad (3.20)$$

and hence is $O(\gamma^2)$.

The proof of Lemma 3.5 is complete provided that we prove the following lemma. \square

Lemma 3.8. *For any positive bounded function g one has*

$$\sup_{\gamma > 0} \mathbb{E} \left[e^{-\int_{\mathbb{U}} g(x) (e^{\gamma X(x)} - 1 - \gamma X(x) - \frac{\gamma^2}{2} \ln C(x, \mathbb{U})) dx} \right] < +\infty.$$

Proof. For notational simplicity we assume in the proof that $g \equiv 1$ and that $\int_{\mathbb{U}} dx = 1$ but the proof with general g (and in particular $g = e^{U(x)}$) works just the same. For the same reasons, we further assume that $C(x, \mathbb{U}) \leq 1$ on the domain. Our strategy is to introduce first the white noise cutoff $(X_\varepsilon)_\varepsilon$ for the field X .

Introducing the cutoff. Recall that $e^{\gamma X_\varepsilon(x)}$ stands for $e^{\gamma X_\varepsilon(x) - \frac{\gamma^2}{2} \mathbb{E}[X_\varepsilon(x)^2]}$. Then as $e^x - x - 1$ is a positive function, we have

$$-\gamma^{-2} \left(C(x, \mathbb{U})^{\frac{\gamma^2}{2}} : e^{\gamma X_\varepsilon(x)} : -1 - \gamma X_\varepsilon(x) - \frac{\gamma^2}{2} \ln C(x, \mathbb{U}) \right) \leq \frac{1}{2} \mathbb{E}[X_\varepsilon^2(x)] \leq \frac{1}{2} |\ln \varepsilon|. \quad (3.21)$$

For the rest of the proof, we use the notation

$$R_\varepsilon(x) := C(x, \mathbb{U})^{\frac{\gamma^2}{2}} : e^{\gamma X_\varepsilon(x)} : -1 - \gamma X_\varepsilon(x) - \frac{\gamma^2}{2} \ln C(x, \mathbb{U}) - \frac{\gamma^2}{2} : X_\varepsilon^2(x) : \quad (3.22)$$

We want to find a good bound for $\int_{\mathbb{U}} |R_\varepsilon(x)| dx$ which holds with a large probability and use (3.21) to bound the exponential when the bound on $\int_{\mathbb{U}} |R_\varepsilon(x)| dx$ is not satisfactory. We set $\varepsilon := e^{-\gamma^{-1/8}}$ and hence $|\ln \varepsilon| = \gamma^{-1/8}$. We stress that this relation will hold during the rest of the proof of lemma 3.8.

We have for any x

$$\mathbb{E} [|R_\varepsilon(x)| \mathbf{1}_{\{|X_\varepsilon(x)| > |\ln \varepsilon|^2\}}] \leq e^{-c|\ln \varepsilon|^3} (1 + |\ln C(x, \mathbb{U})|). \quad (3.23)$$

This inequality can be established without being subtle: use the triangular inequality to decompose $|R_\varepsilon(x)|$ and then estimate each term with standard Gaussian computations.

Separating the space into the two events $\{|X_\varepsilon(x)| > |\ln \varepsilon|^2\}$ and $\{|X_\varepsilon(x)| \leq |\ln \varepsilon|^2\}$ and using the inequality $e^u - 1 - u - u^2/2 \geq u^3/6$ for $u \in \mathbb{R}$ on the second event, we deduce

$$R_\varepsilon(x) \leq |R_\varepsilon(x)| \mathbf{1}_{\{|X_\varepsilon(x)| > |\ln \varepsilon|^2\}} + C\gamma^{9/4} (1 + |\ln C(x, \mathbb{U})|^3). \quad (3.24)$$

Let us set

$$\mathcal{A} := \left\{ \left(\int_{\mathbb{U}} |R_\varepsilon(x)| \mathbf{1}_{\{|X_\varepsilon(x)| > |\ln \varepsilon|^2\}} dx \right) \geq \gamma^3 \right\}.$$

We can integrate the inequality (3.24) with respect to the Lebesgue measure over \mathbb{U} . To bound the integrated first term in the right-hand side of (3.24), we can use (3.23) and the Markov inequality to obtain

$$\mathbb{P}[\mathcal{A}] \leq e^{-c|\ln \varepsilon|^3} \gamma^{-3}. \quad (3.25)$$

We can finally conclude, using (3.21) and (3.24), that

$$\begin{aligned} \mathbb{E} \left[\exp \left(-\frac{1}{\gamma^2} \int_{\mathbb{U}} (C(x, \mathbb{U})^{\frac{\gamma^2}{2}} : e^{\gamma X_\varepsilon(x)} : -1 - \gamma X_\varepsilon(x) - \frac{\gamma^2}{2} \ln C(x, \mathbb{U})) dx \right) \right] \\ \leq e^{C\gamma^{1/4}} \mathbb{E} \left[e^{-\frac{1}{2} \int_{\mathbb{U}} : X_\varepsilon^2 :} \mathbf{1}_{\mathcal{A}^c} \right] + \mathbb{P}[\mathcal{A}] e^{\frac{1}{2} |\ln \varepsilon|}, \end{aligned} \quad (3.26)$$

which can be bounded above by a constant (independent of γ) thanks to the bound

$$\sup_{\varepsilon > 0} \mathbb{E} \left[e^{-\frac{1}{2} \int_{\mathbb{U}} : X_\varepsilon^2 :} \right] < +\infty$$

and (3.25).

Removing the cutoff. The first observation is that we have the following estimate for any event \mathcal{B} by using the Cauchy-Schwarz inequality (and the fact that the exponential is positive and $C(x, \mathbb{U}) \leq 1$)

$$\mathbb{E} \left[e^{-\frac{1}{\gamma^2} \int_{\mathbb{U}} (e^{\gamma X(x)} - 1 - \gamma X(x) - \frac{\gamma^2}{2} \ln C(x, \mathbb{U})) dx} \mathbf{1}_{\mathcal{B}} \right] \leq e^{\gamma^{-2}} \mathbb{E} \left[e^{\frac{1}{\gamma} \int_{\mathbb{U}} X(x) dx} \mathbf{1}_{\mathcal{B}} \right] \leq e^{C\gamma^{-2}} \sqrt{\mathbb{P}[\mathcal{B}]}, \quad (3.27)$$

where C is a positive constant. Now we set $M = \int_{\mathbb{U}} :e^{\gamma X(x)}: dx$, $M_\varepsilon = \int_{\mathbb{U}} C(x, \mathbb{U})^{\gamma^2/2} :e^{\gamma X_\varepsilon(x)}: dx$. On the event \mathcal{B}^c , we can write

$$\begin{aligned} & \mathbb{E} \left[e^{-\frac{1}{\gamma^2} \int_{\mathbb{U}} (e^{\gamma X(x)} - 1 - \gamma X(x) - \frac{\gamma^2}{2} \ln C(x, \mathbb{U})) dx} \mathbf{1}_{\mathcal{B}^c} \right] \\ &= \mathbb{E} \left[e^{-\frac{1}{\gamma^2} \int_{\mathbb{U}} (C(x, \mathbb{U})^{\gamma^2/2} :e^{\gamma X_\varepsilon(x)}: - 1 - \gamma X_\varepsilon(x) - \frac{\gamma^2}{2} \ln C(x, \mathbb{U})) dx} e^{\frac{1}{\gamma} \int_{\mathbb{U}} (X - X_\varepsilon)(x) dx} e^{\frac{1}{\gamma^2} (M_\varepsilon - M)} \mathbf{1}_{\mathcal{B}^c} \right]. \end{aligned} \quad (3.28)$$

It is therefore relevant to consider an event \mathcal{B} such that we can properly estimate the last two exponential terms. A reasonable choice is to set

$$\mathcal{B} := \left\{ \left| \int_{\mathbb{U}} (X - X_\varepsilon)(x) dx \right| \geq \gamma^2 \right\} \cup \left\{ (M_\varepsilon - M) \geq \gamma^3 \right\}.$$

On the event \mathcal{B}^c , (3.28) allows us to compare (with constants) the desired quantity with the cutoff version.

To use (3.27) on the event \mathcal{B} , what remains to do is to prove that

$$\mathbb{P}(\mathcal{B}) \leq e^{-3C\gamma^{-2}}. \quad (3.29)$$

The quantity $\int_{\mathbb{U}} (X - X_\varepsilon)(x) dx$ is a Gaussian random variable whose variance is of order ε . Hence there exists a constant c such that

$$\mathbb{P} \left[\int_{\mathbb{U}} (X - X_\varepsilon)(x) dx \geq \gamma^2 \right] \leq \exp(-c\gamma^4 \varepsilon^{-1}) \quad (3.30)$$

To evaluate the likeliness of a deviation of $M_\varepsilon - M$, we are going to compute the exponential moment of this variable with respect to $\mathbb{E}^\varepsilon := \mathbb{E}[\cdot | \mathcal{F}_\varepsilon]$ where \mathcal{F}_ε is the sigma-algebra generated by the random variables $\{X_u(x); \varepsilon \leq u, x \in \mathbb{U}\}$. For $t > 0$ let us consider the function

$$\phi(t) = \mathbb{E} \left[e^{t(M_\varepsilon - M)} \mid \mathcal{F}_\varepsilon \right] < \infty. \quad (3.31)$$

Note that neither the full expectation with respect to \mathbb{P} nor the expectation for negative t are finite. We have

$$\phi'(t) = \mathbb{E}^\varepsilon \left[(M_\varepsilon - M) e^{t(M_\varepsilon - M)} \right] = \int_{\mathbb{U}} C(x, \mathbb{U})^{\gamma^2/2} :e^{\gamma X_\varepsilon(x)}: \mathbb{E}^\varepsilon \left[e^{t(M_\varepsilon - M)} - e^{t(M_\varepsilon - \tilde{M}^x)} \right] dx \quad (3.32)$$

where

$$\tilde{M}^x := \int_{\mathbb{U}} e^{\gamma X(y)} e^{\gamma^2 \tilde{G}_\varepsilon(x, y)} dy, \quad (3.33)$$

and $\tilde{G}_\varepsilon(x, y) > 0$ is the correlation function of $X - X_\varepsilon$. Note that for any $x \in \mathbb{U}$

$$e^{t(M - \tilde{M}^x)} = \exp \left(-t \int_{\mathbb{U}} (e^{\gamma^2 \tilde{G}_\varepsilon(x, y)} - 1) e^{\gamma X(y)} dy \right)$$

and e^{-tM} are decreasing functions of the field $X - X_\varepsilon$. Hence making use of the FKG inequality for white noise (see [32, section 2.2] for the case of countable product and note that by expression (2.4) the field $X - X_\varepsilon$ is an increasing function of the white noise) for the field $X - X_\varepsilon$ and the inequality $e^u \geq 1 + u$

$$\begin{aligned} \mathbb{E}^\varepsilon \left[e^{-t\tilde{M}^x} \right] &\geq \mathbb{E}^\varepsilon \left[e^{t(M - \tilde{M}^x)} \right] \mathbb{E}^\varepsilon \left[e^{-tM} \right] \\ &\geq \mathbb{E}^\varepsilon \left[1 + t(M - \tilde{M}^x) \right] \mathbb{E}^\varepsilon \left[e^{-tM} \right] \\ &= \left(1 - t \int_{\mathbb{U}} \left[e^{\gamma^2 \tilde{G}_\varepsilon(x, y)} - 1 \right] C(x, \mathbb{U})^{\gamma^2/2} :e^{\gamma X_\varepsilon(y)}: dy \right) \mathbb{E}^\varepsilon \left[e^{-tM} \right]. \end{aligned}$$

Combining this with (3.32) and using that $C(x, \mathbb{U}) \leq 1$ and $e^{\gamma X_\varepsilon(y)} \leq e^{\gamma X_\varepsilon}$, we obtain

$$\phi'(t) \leq t\phi(t) \iint_{\mathbb{U}^2} \left[e^{\gamma^2 \bar{G}_\varepsilon(x,y)} - 1 \right] e^{\gamma X_\varepsilon(x) + X_\varepsilon(y)} dx dy. \quad (3.34)$$

With this we can conclude that

$$\phi(t) \leq e^{\frac{Z_\varepsilon t^2}{2}}, \quad (3.35)$$

where

$$Z_\varepsilon := \iint_{\mathbb{U}^2} \left[e^{\gamma^2 \bar{G}_\varepsilon(x,y)} - 1 \right] e^{\gamma X_\varepsilon(x) + \gamma X_\varepsilon(y)} dx dy.$$

The last thing that we need is a good control on Z_ε . Note that one can find a constant C such that for all $x \in \mathbb{U}$, for all $\gamma < 1$, and ε

$$\int_{\mathbb{U}} \left[e^{\gamma^2 \bar{G}_\varepsilon(x,y)} - 1 \right] dy \leq C\gamma^2 \varepsilon. \quad (3.36)$$

Hence using the inequality

$$e^{\gamma X_\varepsilon(x) + X_\varepsilon(y)} \leq \frac{1}{2} \left(e^{2\gamma X_\varepsilon(x)} + e^{2\gamma X_\varepsilon(y)} \right), \quad (3.37)$$

and symmetries in the integration we obtain

$$Z_\varepsilon \leq \iint_{\mathbb{U}^2} \left[e^{\gamma^2 \bar{G}_\varepsilon(x,y)} - 1 \right] e^{2\gamma X_\varepsilon(x)} dy dx \leq C\gamma^2 \varepsilon \int_{\mathbb{U}} e^{2\gamma X_\varepsilon(x)} dx. \quad (3.38)$$

Now we have

$$\int_{\mathbb{U}} e^{2\gamma X_\varepsilon(x)} dx = \varepsilon^{-1/2} + \int_{\mathbb{U}} e^{2\gamma X_\varepsilon(x)} \mathbf{1}_{\{\gamma X_\varepsilon(x) \geq |\ln \varepsilon|/4\}} dx \quad (3.39)$$

and one can find $c > 0$ such that

$$\mathbb{E} \left[\int_{\mathbb{U}} e^{2\gamma X_\varepsilon(x)} \mathbf{1}_{\{\gamma X_\varepsilon(x) \geq |\ln \varepsilon|/4\}} dx \right] \leq e^{-c\gamma^{-2} |\ln \varepsilon|}. \quad (3.40)$$

Hence using the Markov property (changing the value of c if needed) we have for γ small enough

$$\mathbb{P} [Z_\varepsilon \geq \sqrt{\varepsilon}] \leq e^{-c\gamma^{-2} |\ln \varepsilon|}. \quad (3.41)$$

Finally using (3.35) with $t = \varepsilon^{-1/4}$ we have that

$$\mathbb{P} \left[(M_\varepsilon - M) \geq \gamma^3 \mid Z_\varepsilon \leq \varepsilon^{1/2} \right] \leq \exp(-\varepsilon^{-1/4} \gamma^3) e^{1/2}.$$

This entails that

$$\mathbb{P} [(M_\varepsilon - M) \geq \gamma^3] \leq e^{-c\gamma^{-2} |\ln \varepsilon|} + 2 \exp(-\varepsilon^{-1/4} \gamma^3) e^{1/2}.$$

which completes the proof since we have $\varepsilon = e^{-\gamma^{-1/8}}$. \square

3.4 Proof of the large deviation principle

Proof of Proposition 3.3. The fact that I^* is a good rate function will follow from Theorem 3.4. So we focus on establishing the expression of I^* on $H_0^1(\mathbb{U})$. Our strategy is to first establish the identity on a dense subset of $H_0^1(\mathbb{U})$ and then use a bit of topology to extend it.

For $h \in H^{-1}(\mathbb{U})$, let us denote by H the mapping

$$H : f \in H_0^1(\mathbb{U}) \mapsto h(f) - (\mathbb{F}(\Lambda, f) - \mathbb{F}(\Lambda)).$$

As it is Gâteaux-differentiable (see Proposition A.2), we can compute the partial derivative evaluated at f in the direction $v \in H_0^1(\mathbb{U})$, call it $\partial_v H(f)$. To this purpose, let us introduce the solution V of (3.6) and the function $g \in H_0^1(\mathbb{U})$ solution of (A.4) with h in (A.4) replaced by v . From Proposition A.2 and the definition of \mathbb{F} .

$$\begin{aligned} \partial_v H(f) &= \lim_{t \rightarrow \infty} \frac{H(f + tv) - H(f)}{t} \\ &= h(v) + \frac{1}{4\pi} \int_{\mathbb{U}} (2\langle \partial V, \partial g \rangle + 16\pi^2 \Lambda e^V g - 4\pi f g) dx - \int_{\mathbb{U}} v(V - U) dx \\ &= h(v) - \int_{\mathbb{U}} v(V - U) dx. \end{aligned} \quad (3.42)$$

To get the last line, we have used the fact that V is the solution of (3.6) in such a way that the first integral in the first line vanishes. Let us define

$$\mathcal{S} = \{h \in H_0^1(\mathbb{U}); h = V - U; V \text{ solution to (3.6) for some function } f \in H_0^1(\mathbb{U})\}. \quad (3.43)$$

If $h \in \mathcal{S}$, we can choose f such that $h = V - U$ where V is a solution to (3.6). Then, by (3.42), we deduce that f is a critical point of H , which is concave. Therefore $I^*(h) = H(f)$. Plugging the relation $h = V - U$ into the expression of $H(f)$, we get $I^*(h) = E(V) - E(U) = E(U + h) - E(U)$. This provides the expression of I^* on \mathcal{S} .

Now we show that \mathcal{S} is dense set in $H_0^1(\mathbb{U})$. For this purpose we show that

$$\mathcal{S} = \{h \in H_0^1(\mathbb{U}) \mid \Delta h \in H_0^1(\mathbb{U})\}. \quad (3.44)$$

Indeed if one sets $V = U + h \in H_0^1(\mathbb{U})$. We have

$$\begin{aligned} \Delta V &= \Delta U + \Delta h \\ &= 8\pi^2 \Lambda e^V + (\Delta U + \Delta h - 8\pi^2 \Lambda e^V) \\ &= 8\pi^2 \Lambda e^V + (8\pi^2 \Lambda e^U + \Delta h - 8\pi^2 \Lambda e^V). \end{aligned}$$

Setting $f = (4\pi \Lambda e^V - 4\pi \Lambda e^U - (2\pi)^{-1} \Delta h)$, it remains to prove that f belongs to $H_0^1(\mathbb{U})$ if and only if Δh does. In both cases it suffices to prove that $e^U - e^V \in H_0^1(\mathbb{U})$ but this is easy: because it vanishes on the boundary because h does, and $\partial(e^U - e^V) = e^U \partial U - e^V \partial V$ is square integrable. For this last point, one can check that either $\Delta h \in H_0^1(\mathbb{U})$ (easy) or $f \in H_0^1(\mathbb{U})$ (using $\Delta V = 8\pi^2 \Lambda e^V - 2\pi f$) implies that V is bounded.

Now we establish the expression of I^* on $H_0^1(\mathbb{U}) \setminus \mathcal{S}$ by density. Note that as a supremum of continuous linear functions, I^* restricted to $H_0^1(\mathbb{U})$ is weakly lower semi-continuous (for the $H_0^1(\mathbb{U})$ norm). By continuity of $E(\cdot)$ for the $H_0^1(\mathbb{U})$ norm, approximating $h \in H_0^1$ by a sequence in \mathcal{S} we deduce that

$$\forall h \in H_0^1(\mathbb{U}), \quad I^*(h) \leq E(U + h) - E(U). \quad (3.45)$$

Conversely, if $h \in H_0^1(\mathbb{U})$, we can find a sequence $(h_n)_n$ in \mathcal{S} converging in $H_0^1(\mathbb{U})$ towards h . For each n , let us consider $f_n \in H_0^1(\mathbb{U})$ such that $h_n = V_n - U$ where V_n is the solution to (3.6) associated to f_n . From (3.6), one can check that $(f_n)_n$ is Cauchy and strongly converges in $H^{-1}(\mathbb{U})$. Then we get:

$$\begin{aligned} I^*(h) &= \sup_{f \in H_0^1(\mathbb{U})} h(f) - F(\Lambda, f) + F(\Lambda) \\ &\geq h(f_n) - F(\Lambda, f_n) + F(\Lambda) \\ &= (h - h_n)(f_n) + h_n(f_n) - F(\Lambda, f_n) + F(\Lambda) \\ &= (h - h_n)(f_n) + E(U + h_n) - E(U). \end{aligned}$$

We conclude by observing that $(h - h_n)(f_n) \rightarrow 0$ and $E(U + h_n) - E(U) \rightarrow E(U + h) - E(U)$ as $n \rightarrow \infty$.

To complete the proof, we show that $I^*(h) < +\infty$ implies $h \in H_0^1(\mathbb{U})$. For each $\bar{f} \in H_0^1(\mathbb{U})$, let us consider the associated solution to (3.6) and define $\bar{h} = \bar{V} - U$. Repeating the above argument, we have for each $\bar{f} \in H_0^1(\mathbb{U})$

$$I^*(h) \geq (h - \bar{h})(\bar{f}) + E(U + \bar{h}) - E(U). \quad (3.46)$$

As $I^*(h) < +\infty$ and $E(U + \bar{h}) - E(U) \geq 0$, we deduce that

$$\forall \bar{f} \in H_0^1(\mathbb{U}), \quad h(\bar{f}) \leq C + \bar{h}(\bar{f}), \quad (3.47)$$

for some constant $C > 0$, which does not depend on \bar{f} . Let us further introduce a function $\bar{g} \in H_0^1(\mathbb{U})$ such that $-2\pi\bar{f} = \Delta\bar{g}$. To establish that $h \in H_0^1(\mathbb{U})$, it suffices to prove that the above right-hand side of (3.47) is bounded uniformly when $\|\bar{f}\|_{H^{-1}} \leq 1$ (or equivalently $\|\bar{g}\|_{H^1} \leq 1$). By integrating (3.6) with respect to \bar{V} , we get

$$-\int_{\mathbb{U}} |\partial\bar{V}|^2 dx = 8\pi^2\Lambda \int_{\mathbb{U}} e^{\bar{V}} \bar{V} dx + \int_{\mathbb{U}} \Delta\bar{g} \bar{V} dx,$$

which can be rewritten as

$$\int_{\mathbb{U}} |\partial\bar{V}|^2 dx + 8\pi^2\Lambda \int_{\mathbb{U}} (e^{\bar{V}} - 1) \bar{V} dx = -8\pi^2\Lambda \int_{\mathbb{U}} \bar{V} dx + \int_{\mathbb{U}} \langle \partial\bar{g}, \partial\bar{V} \rangle dx. \quad (3.48)$$

Using the elementary inequality $\langle a, b \rangle \leq \frac{1}{2c}|a|^2 + \frac{c}{2}|b|^2$ for the two terms for a well chosen $c > 0$ we can establish that the right-hand side is less than $C' + \frac{1}{2} \int_{\mathbb{U}} |\partial\bar{V}|^2 dx$, for some constant C' that does not depend on \bar{f} . Observe that $(e^u - 1)u \geq 0$ for all $u \in \mathbb{R}$ so we deduce that $\int_{\mathbb{U}} |\partial\bar{V}|^2 dx$ is bounded uniformly on the set $\{\bar{f} \in H_0^1(\mathbb{U}) \mid \|\bar{f}\|_{H^{-1}} \leq 1\}$ (and thus $\int_{\mathbb{U}} |\partial\bar{h}|^2 dx$ too). Finally, we have

$$\bar{h}(\bar{f}) = \int_{\mathbb{U}} \langle \partial\bar{g}, \partial\bar{h} \rangle dx \leq |\bar{g}|_{H^1} |\bar{h}|_{H^1}$$

so that $\bar{h}(\bar{f})$ is uniformly bounded on the set $\{\bar{f} \in H_0^1(\mathbb{U}) \mid \|\bar{f}\|_{H^{-1}} \leq 1\}$. This implies that $h \in H_0^1(\mathbb{U})$.

Also, recall that U is the unique minimum in $H_0^1(\mathbb{U})$ of the functional E . Indeed, a function in $H_0^1(\mathbb{U})$ is a minimum of this functional if and only if it is a weak solution to (3.3). Furthermore, the weak solution of (3.3) is unique as we have proved that the field γX converges in law (and even in probability) towards U as soon as we get a weak solution U to this equation. The limit

in law being unique, we get uniqueness for (3.3). In particular, if $h \in H_0^1(\mathbb{U})$ and $h \neq 0$, we get $I^*(h) = E(U + h) - E(U) > 0$. \square

Proof of Theorem 3.4. Assume that we can prove that the family $(Y_\gamma)_\gamma$ is exponentially tight and that for each function $f \in H_0^1(\mathbb{U})$

$$\lim_{\gamma \rightarrow 0} \gamma^2 \ln \mathbb{E}_{\mu, \gamma} \left[e^{\frac{Y_\gamma(f)}{\gamma^2}} \right] = F(\Lambda, f) - F(\Lambda). \quad (3.49)$$

The mapping $f \in H_0^1(\mathbb{U}) \mapsto F(\Lambda, f) - F(\Lambda)$ is Gâteaux-differentiable as shown in (3.42) and weakly lower semi-continuous (even weakly continuous) from Proposition A.3. Hence we can apply a standard result from the theory of Large deviation in functional spaces [17, Corollary 4.5.27], which entails the proof of Theorem 3.4.

So we focus on establishing (3.49) first and then we will prove that the family $(Y_\gamma)_\gamma$ is exponentially tight. As we already know the asymptotic behavior of the partition function, it is sufficient to compute the asymptotic behavior of

$$Z_{\mu, \gamma} \left[e^{\frac{Y_\gamma(f)}{\gamma^2}} \right] = \mathbb{E} \left[\exp \left(\frac{Y_\gamma(f)}{\gamma^2} - \frac{4\pi\Lambda}{\gamma^2} \int_{\mathbb{U}} e^{\gamma X(x)} dx \right) \right].$$

Let V be the (deterministic) weak solution of (3.6) and set

$$\theta(x) := f(x) - 4\pi\Lambda e^{V(x)} = -\frac{\Delta V(x)}{2\pi}. \quad (3.50)$$

Note that it implies

$$\int_{\mathbb{U}} \theta(y) G_{\mathbb{U}}(x, y) dy = V(x) \quad (3.51)$$

We define H_γ to be a shifted version of the field X ,

$$H_\gamma(x) = X - \frac{V}{\gamma}. \quad (3.52)$$

We have

$$\begin{aligned} & \mathbb{E} \left[\exp \left(\frac{Y_\gamma(f)}{\gamma^2} - \frac{4\pi\Lambda}{\gamma^2} \int_{\mathbb{U}} e^{\gamma X(x)} dx \right) \right] \\ &= \mathbb{E} \left[e^{\frac{1}{\gamma^2} \int_{\mathbb{U}} (\gamma X(x) - U(x)) f(x) dx - \frac{4\pi\Lambda}{\gamma^2} \int_{\mathbb{U}} e^{V(x)} (1 + \gamma H_\gamma(x)) dx} e^{-\frac{4\pi\Lambda}{\gamma^2} \int_{\mathbb{U}} e^{\gamma X(x) - V(x)} (1 + \gamma H_\gamma(x)) dx} \right] \\ &= e^{\frac{1}{\gamma^2} \int_{\mathbb{U}} (4\pi\Lambda V(x) e^{V(x)} - 4\pi\Lambda e^{V(x)} - U(x) f(x)) dx + \frac{1}{2\gamma^2} \iint_{\mathbb{U}^2} \theta(x) \theta(y) G_{\mathbb{U}}(x, y) dx dy} \\ & \times \mathbb{E} \left[e^{\frac{1}{\gamma} \int_{\mathbb{U}} X(x) \theta(x) dx - \frac{1}{2\gamma^2} \iint_{\mathbb{U}^2} \theta(x) \theta(y) G_{\mathbb{U}}(x, y) dx dy} e^{-\frac{4\pi\Lambda}{\gamma^2} \int_{\mathbb{U}} e^{\gamma X(x) - V(x)} (1 + \gamma H_\gamma(x)) dx} \right]. \end{aligned} \quad (3.54)$$

Once again, the first exponential term in the expectation

$$e^{\frac{1}{\gamma} \int_{\mathbb{U}} X(x) \theta(x) dx - \frac{1}{2\gamma^2} \iint_{\mathbb{U}^2} \theta(x) \theta(y) G_{\mathbb{U}}(x, y) dx dy}$$

is a Girsanov transform term. It has the effect of shifting the field X by an amount $\gamma^{-1}V$ (cf. (3.51)), and hence after this shift, H_γ becomes a centered field, and the expectation in the last line of (3.54) is equal to

$$\mathbb{E} \left[\exp \left(-\frac{4\pi\Lambda}{\gamma^2} \int_{\mathbb{U}} e^{V(x)} (e^{\gamma X(x)} - 1 - \gamma X(x)) dx \right) \right].$$

Concerning the exponential term in front of the expectation, it can be simplified. Let us briefly explain how. From (3.51) and (3.50)

$$\iint_{\mathbb{U}^2} \theta(x)\theta(y)G_{\mathbb{U}}(x,y)dxdy = \int_{\mathbb{U}} \theta(x)V(x)dx = -\frac{1}{2\pi} \int_{\mathbb{U}} \Delta V(x)V(x)dx = \frac{1}{2\pi} \int_{\mathbb{U}} |\partial V(x)|^2 dx. \quad (3.55)$$

We also have

$$\int_{\mathbb{U}} 4\pi\Lambda e^{V(x)}V(x)dx = \int_{\mathbb{U}} (f(x) - \theta(x))V(x) = -\frac{1}{2\pi} \int_{\mathbb{U}} |\partial V(x)|^2 dx + \int_{\mathbb{U}} f(x)V(x). \quad (3.56)$$

Using this in (3.54) we obtain

$$\begin{aligned} & \mathbb{E}\left[\exp\left(\frac{Y_{\gamma}(f)}{\gamma^2} - \frac{4\pi\Lambda}{\gamma^2} \int_{\mathbb{U}} e^{\gamma X(x)} dx\right)\right] \\ &= \exp\left(-\frac{1}{4\pi\gamma^2} \int_{\mathbb{U}} (|\partial V(x)|^2 + 16\pi^2\Lambda e^{V(x)}) + \frac{1}{\gamma^2} \int_{\mathbb{U}} f(x)(V(x) - U(x))dx\right) \\ & \quad \times \mathbb{E}\left[\exp\left(-\frac{4\pi\Lambda}{\gamma^2} \int_{\mathbb{U}} e^{V(x)}(e^{\gamma X(x)} - 1 - \gamma X(x))dx\right)\right]. \end{aligned} \quad (3.57)$$

To complete the proof of (3.49): we use Lemma 3.5 which asserts that the last line converges as $\gamma \rightarrow 0$ towards

$$e^{-2\pi\Lambda \int_{\mathbb{U}} e^{V(x)} \ln C(x, \mathbb{U}) dx} \mathbb{E}[e^{-2\pi\Lambda \int_{\mathbb{U}} e^{V(x)} :X^2(x): dx}].$$

Now, we turn to the exponential tightness of the field $Y_{\gamma} = \gamma X - U$. The exponential tightness of the field $Y_{\gamma} = \gamma X - U$ is equivalent to the exponential tightness of γX (simply because if K is a compact of $H^{-1}(\mathbb{U})$, $K + U$ is also a compact). We adopt the framework of section 4.2 in [22]. By conformal invariance, we work on the square $S = [0, 1]^2$. In this case, given a sequence $(a_{j,k})_{j,k \geq 1}$, the series

$$f_n := \sum_{1 \leq j,k \leq n} a_{j,k} \sin(\pi j x) \sin(\pi j y) \quad (3.58)$$

converges in $H^{-1}(S)$ if and only if $\sum_{j,k \geq 1} \frac{|a_{j,k}|^2}{j^2 + k^2} < \infty$. In this case the limit $f := \lim_n f_n$ has the following norm

$$|f|_{H^{-1}(S)} = \sum_{j,k \geq 1} \frac{|a_{j,k}|^2}{(j^2 + k^2)}.$$

In $H^{-1}(S)$, the GFF is then the almost sure limit of the series (3.58) where $a_{j,k} = \frac{\varepsilon_{j,k}}{\sqrt{j^2 + k^2}}$ where $(\varepsilon_{j,k})_{j,k \geq 1}$ is an i.i.d. sequence of standard Gaussian variables (in this case the $\varepsilon_{j,k}$ are the $H^1(S)$ projections of X on the $H^1(S)$ basis $((x, y) \rightarrow \sin(\pi j x) \sin(\pi j y))_{j,k \geq 1}$). Let $C > 0$ be fixed. We introduce the following compact set of $H^{-1}(S)$ (we identify the limit of the series (3.58) with the sequence $(a_{j,k})_{j,k \geq 1}$)

$$K_C = \{(a_{j,k})_{j,k \geq 1}; \forall j, k, |a_{j,k}| \leq \frac{C}{(j^2 + k^2)^{1/4}}\}$$

We have

$$\begin{aligned} \mathbb{P}(\gamma X \notin K_C) &= \mathbb{P}(\exists j, k, \gamma |\varepsilon_{j,k}| > C(j^2 + k^2)^{1/4}) \\ &\leq \sum_{j,k \geq 1} \mathbb{P}(\gamma |\varepsilon_{j,k}| > C(j^2 + k^2)^{1/4}) \\ &\leq \sum_{j,k \geq 1} e^{-\frac{C^2((j^2 + k^2)^{1/2})}{2\gamma^2}}. \end{aligned}$$

hence we get that

$$\lim_{C \rightarrow \infty} \overline{\lim}_{\gamma \rightarrow 0} \gamma^2 \ln \mathbb{P}(\gamma X \notin K_C) = -\infty.$$

This shows that γX is exponentially tight in $H^{-1}(S)$. □

4 Semiclassical limit of LFT with heavy matter insertions

In this section, we want to treat the case of heavy matter operator insertions in the partition function. This roughly corresponds to tilting the partition function of LFT with exponential terms and we will see that, semiclassically, this creates conical singularities in a hyperbolic surface. We restrict once again to the flat unit disk for simplicity.

More precisely, we consider distinct $z_1, \dots, z_p \in \mathbb{U}$, $\chi_1, \dots, \chi_p \in [0, 2]$, a cosmological constant $\mu \geq 0$ and a Liouville conformal factor $\gamma \in]0, 2]$. We set

$$Q = \frac{2}{\gamma} + \frac{\gamma}{2}.$$

We formally define the law $\mathbb{P}_{\mu, \gamma, (z_i, \chi_i)_i}$ of the Liouville field X on \mathbb{U} with heavy matter insertions $(z_i, \chi_i)_i$ associated to (μ, γ) as the law of the GFF on \mathbb{U} tilted by

$$\exp \left(-4\pi\mu \int_{\mathbb{U}} e^{\gamma X(x)} dx \right) \prod_{i=1}^p e^{\frac{\chi_i}{\gamma} X(z_i)}, \quad (4.1)$$

Of course, the above expression is not a function (because of $e^{\frac{\chi_i}{\gamma} X(z_i)}$) and this cannot be considered as a Radon-Nykodym derivative, but on a formal level one can always consider this last term as a Girsanov tilt. The rigorous definition of $\mathbb{P}_{\mu, \gamma, (z_i, \chi_i)_i}$ is then given by its action on bounded continuous functionals F on $H^{-1}(\mathbb{U})$ as follows

$$\begin{aligned} & \mathbb{E}_{\mu, \gamma, (z_i, \chi_i)_i} [F(X)] \\ &= Z_{\mu, \gamma, (z_i, \chi_i)_i}^{-1} \mathbb{E} \left[F \left(X + \sum_i \frac{\chi_i}{\gamma} G_{\mathbb{U}}(\cdot, z_i) \right) \exp \left(-4\pi\mu \int_{\mathbb{U}} e^{\gamma X(x)} e^{\sum_i \chi_i G_{\mathbb{U}}(\cdot, z_i)} dx \right) \right] \end{aligned}$$

where \mathbb{E} stands for the expectation with respect to the free field X , and

$$Z_{\mu, \gamma, (z_i, \chi_i)_i} = \mathbb{E} \left[\exp \left(-4\pi\mu \int_{\mathbb{U}} e^{\gamma X(x)} e^{\sum_i \chi_i G_{\mathbb{U}}(\cdot, z_i)} dx \right) \right]. \quad (4.2)$$

The additional exponential terms in the above product are called heavy matter operators in the physics literature (see [41, 34] for instance). The problem is to compute the asymptotic behaviour of the partition function and to find the limit in law under the probability law $\mathbb{P}_{\mu, \gamma, (z_i, \chi_i)_i}$ of the field γX when $\gamma^2 \mu = \Lambda$ and $\gamma \rightarrow 0$.

We will see that the field concentrates on the solutions of the Liouville equation with sources (see Theorem A.4)

$$\Delta U = 8\pi^2 \Lambda e^U - 2\pi \sum_i \chi_i \delta_{z_i} \quad U|_{\partial \mathbb{U}} = 0, \quad (4.3)$$

where δ_z stands for the Dirac mass at z . Theorem A.4 shows that if U is the solution of equation (4.3) then $U - \sum_i \chi_i G_{\mathbb{U}}(\cdot, z_i)$ is at least continuous. Therefore the metric $e^{U(x)} dx^2$ possesses singularities of the type $\frac{1}{|x - z_i|^{\chi_i}}$ at the points z_i .

Theorem 4.1. Assume $\gamma \rightarrow 0$ while keeping fixed the quantity $\gamma^2\mu = \Lambda$. The field γX concentrates on the solution of the Liouville equation with sources (4.3). More precisely

1. The partition function has the following asymptotic behavior at the exponential scale

$$\lim_{\gamma \rightarrow 0} \gamma^2 \ln Z_{\mu, \gamma, (z_i, \chi_i)_i} = -\frac{1}{4\pi\gamma^2} \int_{\mathbb{U}} (|\partial(U - H)(x)|^2 + 16\pi^2 \Lambda e^{U(x)}) dx =: F(\Lambda, (z_i, \chi_i)_i). \quad (4.4)$$

where $H(x) = \sum_i \chi_i G_{\mathbb{U}}(\cdot, z_i)$.

2. More precisely we have the following equivalent as $\gamma \rightarrow 0$

$$Z_{\mu, \gamma, (z_i, \chi_i)_i} \sim e^{\gamma^{-2} F(\Lambda, (z_i, \chi_i)_i)} \exp \left(-2\pi\Lambda \int_{\mathbb{U}} e^{U(x)} \ln C(x, \mathbb{U}) dx \right) \times \mathbb{E} \left[\exp \left(-2\pi\Lambda \int_{\mathbb{U}} e^{U(x)} : X(x)^2 : dx \right) \right]. \quad (4.5)$$

3. The field γX converges in probability in $H^{-1}(\mathbb{U})$ as $\gamma \rightarrow 0$ towards U .
4. Both random measures $: e^{\gamma X} : dx$ and $e^{\gamma X} dx$ converge in law in the sense of weak convergence of measures towards $e^{U(x)} dx$ as $\gamma \rightarrow 0$.
5. the field $X - \gamma^{-1}U$ converges in law in $H^{-1}(\mathbb{U})$ towards a Massive Free Field in the metric $\hat{g} = e^{U(x)} dx^2$ with Dirichlet boundary condition and mass $8\pi^2\Lambda$, that is a Gaussian field with covariance kernel given by the Green function of the operator $2\pi(8\pi^2\Lambda - \Delta_{\hat{g}})^{-1}$ and Dirichlet boundary condition.

4.1 The large deviation principle for LFT with insertions

For $f \in H_0^1(\mathbb{U})$, we consider the weak solution V of the perturbed Liouville equation (see Theorem A.1)

$$\Delta V = 8\pi^2 \Lambda e^{V(x)} - 2\pi f(x) - 2\pi \sum_i \chi_i \delta_{z_i}, \quad \text{with } V|_{\partial\mathbb{U}} = 0, \quad (4.6)$$

and we set

$$F(\Lambda, f) = -\frac{1}{4\pi} \int_{\mathbb{U}} (|\partial(V - H)|^2 + 16\pi^2 \Lambda e^{V(x)}) dx + \int_{\mathbb{U}} (V - U - H)(x) f(x) dx,$$

where U is the solution of the classical Liouville equation (4.3). The mapping $f \in H_0^1(\mathbb{U}) \mapsto F(\Lambda, f) - F(\Lambda)$ is still convex, Gâteaux-differentiable and weakly lower semi-continuous.

We define the Fenchel-Legendre transform I^* of $F(\Lambda, \cdot) - F(\Lambda)$ as prescribed by (3.7). We further define the set

$$\mathcal{S}_{\text{source}} = \{h \in H^{-1}(\mathbb{U}); h = V - U - H; V \text{ solution to (3.6) for some function } f \in H_0^1(\mathbb{U})\}.$$

The function I^* is a good rate function, with $I^*(h) > 0$ except for $h = 0$. For $h \in \mathcal{S}_{\text{source}}$, we have the following explicit expression

$$I^*(h) = E(V) - E(U) < +\infty, \quad \text{if } h = V - U - H \text{ where } V \text{ solves (4.6) for some } \in H_0^1(\mathbb{U}),$$

and

$$\forall u \in H_0^1(\mathbb{U}) + H, \quad E(u) = \frac{1}{4\pi} \int_{\mathbb{U}} (|\partial(u - H)(x)|^2 + 16\pi^2 \Lambda e^{u(x)}) dx.$$

Finally, $\mathcal{S}_{\text{source}} + H$ is dense in $H^{-1}(\mathbb{U})$ as it contains the set of h such that $h, \Delta h \in H_0^1(\mathbb{U})$.

Theorem 4.2. Assume that $\gamma \rightarrow 0, \mu \rightarrow \infty$ under the constraint (3.2). Set $Y_\gamma = \gamma X - U$. The following LDP holds with good rate function I^* on the space $H^{-1}(\mathbb{U})$ equipped with the norm $|\cdot|_{H^{-1}}$

$$-\inf_{h \in \tilde{A}} I^*(h) \leq \gamma^2 \liminf_{\gamma \rightarrow 0} \mathbb{P}_{\mu, \gamma, (z_i, \chi_i)_i}(Y_\gamma \in A) \leq \gamma^2 \limsup_{\gamma \rightarrow 0} \mathbb{P}_{\mu, \gamma, (z_i, \chi_i)_i}(Y_\gamma \in A) \leq -\inf_{h \in \tilde{A}} I^*(h)$$

for each Borel subset A of $H^{-1}(\mathbb{U})$.

4.2 Proofs

Proof of Theorem 4.1. We first compute the limit of the partition function $Z_{\mu, \gamma}$.

$$Z_{\mu, \gamma, (z_i, \chi_i)_i} = \mathbb{E} \left[e^{-\frac{4\pi\Lambda}{\gamma^2} \int_{\mathbb{U}} e^{\gamma X(x)} e^{\sum_i \chi_i G_{\mathbb{U}}(\cdot, z_i)} dx} \right] = \mathbb{E} \left[e^{-\frac{4\pi\Lambda}{\gamma^2} \int_{\mathbb{U}} e^{\gamma X(x)} e^{\sum_i \chi_i G_{\mathbb{U}}(\cdot, z_i)} C(x, \mathbb{U})^{\frac{\gamma^2}{2}} dx} \right].$$

Let us set $V = U - H$ where U is the solution of (4.3). Note that V satisfies

$$V(x) = -4\pi\Lambda \int_{\mathbb{U}} e^{U(y)} G_{\mathbb{U}}(x, y) dy. \quad (4.7)$$

Finally we set $Y = Y_\gamma = X - \gamma^{-1}V$. The computation as in (3.10) yields

$$\begin{aligned} Z_{\mu, \gamma, (z_i, \chi_i)_i} &= e^{-\frac{4\pi\Lambda}{\gamma^2} \int_{\mathbb{U}} e^{U(x)} (1-V(x)) dx} e^{\frac{8\pi^2\Lambda^2}{\gamma^2} \iint_{\mathbb{U}^2} e^{U(x)+U(y)} G_{\mathbb{U}}(x, y) dx dy} \\ &\times \mathbb{E} \left[e^{-\frac{4\pi\Lambda}{\gamma} \int_{\mathbb{U}} e^{U(x)} X(x) dx - \frac{8\pi^2\Lambda^2}{\gamma^2} \iint_{\mathbb{U}^2} e^{U(x)+U(y)} G_{\mathbb{U}}(x, y) dx dy} e^{-\frac{4\pi\Lambda}{\gamma^2} \int_{\mathbb{U}} e^{H(x)} e^{\gamma X(x)} - e^{U(x)} (1-\gamma Y(x)) dx} \right] \\ &= e^{-\frac{1}{4\pi\gamma^2} \int_{\mathbb{U}} (|\partial V(x)|^2 + 16\pi^2\Lambda e^{U(x)}) dx} \mathbb{E} \left[e^{\int_{\mathbb{U}} e^{U(x)} (e^{\gamma X(x)} - 1 - \gamma X(x)) dx} \right]. \end{aligned}$$

The last line is obtained by using (4.7) to simplify the first term and by performing a Girsanov transform in the expectation which by (4.7) again has the property of shifting the field X by an amount $\gamma^{-1}V$ and makes Y centered. The computation of the partition function as well as the other statements of Theorem 4.1 are completed if one can show that Lemma 3.5 also holds in the case when U is the solution of (4.3). \square

Proof of Lemma 3.5 for $U(x)$ solution of (4.3). What has to be done is to add a factor $e^{U(x)}$ in front of many terms and check that the proof still works. We have to be a bit careful here because $e^{U(x)}$ is not bounded as it possesses singularities at the points where the mass is added. However as these singularities are integrable this causes no major problem. Let us mention a few modifications that are needed for the proof to work: note that before (3.30), it is not true that $\int_{\mathbb{U}} e^{U(x)} (X - X_\varepsilon) dx$ is of order ε , but we still get a power of ε which is ok. In the rest of the proof we just have to use that $e^{U(x)}$ is integrable. \square

Proof of Lemma 3.7 for $U(x)$ solution of (4.3). What we need to do is to find a base of $\mathbb{L}^2(e^{U(x)} dx^2)$ which when suitably normalized is also a base of $H_0^1(\mathbb{U})$. Then the proof of Lemma 3.7 of the previous section applies.

First note that e^U is in $\mathbb{L}^p(\mathbb{U})$ for some $p > 1$. Hence, one can consider the following Hilbert-Schmidt operator on the space $\mathbb{L}^2(e^{U(x)} dx)$

$$f \mapsto \int_{\mathbb{U}} G_{\mathbb{U}}(\cdot, y) f(y) e^{U(y)} dy$$

This symmetric operator can be diagonalized along an orthonormal (in $\mathbb{L}^2(e^{U(x)} dx)$) sequence $(e_j)_{j \geq 1}$ with associated eigenvalues $(\frac{1}{\lambda_j})_{j \geq 1}$ (decreasing order with repetition to account for multiple eigenvalues). We stress that $\sum_j \lambda_j^{-2} < +\infty$. Therefore we have

$$\frac{e_j(x)}{\lambda_j} = \int_{\mathbb{U}} G_{\mathbb{U}}(x, y) e_j(y) e^{U(y)} dy. \quad (4.8)$$

By using Cauchy-Schwartz, we get

$$|\frac{e_j(x)}{\lambda_j}| \leq \left(\int_{\mathbb{U}} G_{\mathbb{U}}(x, y)^2 e^{U(y)} dy \right)^{1/2} \left(\int_{\mathbb{U}} e_j(y)^2 e^{U(y)} dy \right)^{1/2}.$$

Therefore (4.8) implies that e_j is a continuous bounded function. One can then differentiate the expression (4.8) and see that the sequence $(e_j)_{j \geq 1}$ is in $H_0^1(\mathbb{U})$; it is then standard to check that $(\frac{e_j}{\sqrt{\lambda_j}})_{j \geq 1}$ is an orthonormal sequence in $H_0^1(\mathbb{U})$. In fact, the sequence $(\lambda_j)_j$ is the increasing sequence of eigenvalues of $-(2\pi)^{-1} \Delta_g$ with Dirichlet boundary conditions where g is the metric tensor $e^{U(x)} dx^2$. It remains to show that the sequence $(\frac{e_j}{\sqrt{\lambda_j}})_{j \geq 1}$ is a **basis** of $H_0^1(\mathbb{U})$. Consider a function φ in $H_0^1(\mathbb{U})$ which is orthogonal to every $(e_j)_{j \geq 1}$ in H_0^1 . Then as

$$\int_{\mathbb{U}} \varphi(x) e_j(x) e^{U(x)} dx = -2\pi(\lambda_j)^{-1} \int_{\mathbb{U}} \varphi(x) \Delta e_j(x) dy = 2\pi(\lambda_j)^{-1} \int_{\mathbb{U}} \langle \partial \varphi(x), \partial e_j(x) \rangle dx = 0. \quad (4.9)$$

it is also orthogonal to all the $(e_j)_{j \geq 0}$ as an element of $\mathbb{L}^2(e^{U(x)} dx)$ and thus is equal to zero. \square

Proof of Theorem 4.2. As in the proof of Theorem 3.4, the first task is to compute the Laplace transform of linear forms under the Liouville measure.

Let V be the (deterministic) weak solution of (4.6) (see Corollary A.5) and set

$$T := V - H. \quad (4.10)$$

We also consider

$$\theta(x) := f(x) - 4\pi\Lambda e^{V(x)} \quad (4.11)$$

Note our definition together with (4.6) imply that

$$T(x) := \int_{\mathbb{U}} \theta(y) G_{\mathbb{U}}(x, y) dy \quad (4.12)$$

We define Y_γ to be a shifted version of the field X ,

$$Y_\gamma(x) = X - \frac{T}{\gamma}. \quad (4.13)$$

We have

$$\mathbb{E} \left[\exp \left(\gamma^{-1} X(f) - \frac{4\pi\Lambda}{\gamma^2} \int_{\mathbb{U}} e^{H(x)} e^{\gamma X(x)} dx \right) \right] \quad (4.14)$$

$$\begin{aligned} &= \mathbb{E} \left[e^{\frac{1}{\gamma^2} \int_{\mathbb{U}} \gamma X(x) f(x) dx - \frac{4\pi\Lambda}{\gamma^2} \int_{\mathbb{U}} e^{V(x)} (1 + \gamma Y_\gamma(x)) dx} e^{-\frac{4\pi\Lambda}{\gamma^2} \int_{\mathbb{U}} e^{H(x)} e^{\gamma X(x)} - e^{V(x)} (1 + \gamma Y_\gamma(x)) dx} \right] \\ &= e^{\frac{1}{\gamma^2} \int_{\mathbb{U}} (4\pi\Lambda T(x) e^{V(x)} - 4\pi\Lambda e^{V(x)}) dx + \frac{1}{2\gamma^2} \iint_{\mathbb{U}^2} \theta(x) \theta(y) G_{\mathbb{U}}(x, y) dx dy} \\ &\times \mathbb{E} \left[e^{\frac{1}{\gamma} \int_{\mathbb{U}} X(x) \theta(x) dx - \frac{1}{2\gamma^2} \iint_{\mathbb{U}^2} \theta(x) \theta(y) G_{\mathbb{U}}(x, y) dx dy} e^{-\frac{4\pi\Lambda}{\gamma^2} \int_{\mathbb{U}} e^{H(x)} e^{\gamma X(x)} - e^{V(x)} (1 + \gamma Y_\gamma(x)) dx} \right]. \end{aligned} \quad (4.15)$$

The usual Girsanov tricks helps us to control the last term: the tilt shifts X by an amount T/γ (cf. (4.12)) and has the effect of centering Y . Similarly to (3.55) and (3.56), we have (OK)

$$\iint_{\mathbb{U}^2} \theta(x)\theta(y)G_{\mathbb{U}}(x,y)dxdy = \frac{1}{2\pi} \int_{\mathbb{U}} |\partial V(x) - \partial H(x)|^2 dx. \quad (4.16)$$

and

$$\int_{\mathbb{U}} 4\pi\Lambda e^{V(x)}T(x)dx = -\frac{1}{2\pi} \int_{\mathbb{U}} |\partial V(x) - \partial H(x)|^2 dx + \int_{\mathbb{U}} f(x)(V(x) - H(x)) dx, \quad (4.17)$$

which yields

$$\begin{aligned} & \mathbb{E}\left[\exp\left(\gamma^{-1}X(f) - \frac{4\pi\Lambda}{\gamma^2} \int_{\mathbb{U}} e^{H(x)}e^{\gamma X(x)} dx\right)\right] \\ &= \exp\left(-\frac{1}{4\pi\gamma^2} \int_{\mathbb{U}} (|\partial V(x) - \partial H(x)|^2 + 16\pi^2\Lambda e^{V(x)}) dx + \frac{1}{\gamma^2} \int_{\mathbb{U}} f(x)(V(x) - H(x)) dx\right) \\ & \quad \times \mathbb{E}\left[\exp\left(-\frac{4\pi\Lambda}{\gamma^2} \int_{\mathbb{U}} e^{V(x)}(e^{\gamma X(x)} - 1 - \gamma X(x)) dx\right)\right]. \end{aligned} \quad (4.18)$$

and the last expectation converges towards a constant. This gives the exact asymptotic expression of the Laplace transform at exponential scale.

$$\begin{aligned} & \lim_{\gamma \rightarrow 0} \gamma^2 \ln \mathbb{E}_{\mu, \gamma, (z_i, \chi_i)_i} \left[\exp\left(\gamma^{-2}(\gamma X(f) - \int_{\mathbb{U}} U(x)f(x)dx)\right) \right] = \\ & \quad - \frac{1}{4\pi} \int_{\mathbb{U}} (|\partial V - \partial H|^2 + 16\pi^2\Lambda e^{V(x)}) dx + \int_{\mathbb{U}} (V - U - H)(x)f(x)dx. \end{aligned}$$

We can then complete the proof by following the lines of Theorem 3.4 (use Theorem A.4 and Corollary A.5 to study the rate function). \square

5 Liouville Field Theory on the hyperbolic disk

In this section, we take the occasion to explain how to construct the Liouville theory on a hyperbolic Riemann surface in order to show the interaction between the Liouville potential and the curvature of the background metric. It is thus natural to consider the unit disk \mathbb{U} equipped with the hyperbolic (or Poincaré) scalar tensor

$$\hat{g}_{\mathbb{U}}(x) = \frac{4}{(1 - |x|^2)^2}, \quad (5.1)$$

or any conformal reparametrization of such a metric space. This is the prototype of two dimensional Riemann surface with constant negative Ricci scalar curvature $R_{\hat{g}_{\mathbb{U}}} = -2$. There is a subtlety in the choice of such a background structure: keep in mind that the Liouville quantum field theory is intrinsically of negative curvature and thus well fitted for negatively curved background metrics. Other choices of background structures lead to further complications that we do not want to tackle here. The organization of this section is the following. We first introduce some necessary background about "background metrics" and "Free Fields in curved spaces". Then we will introduce the interaction term of LFT. This will allow us to define the law of the Liouville field and the associated random metrics. Finally we discuss some properties of such a theory like the conformal invariance and the KPZ formulae.

5.1 Background metrics

Let us consider a domain D of \mathbb{R}^2 equipped with a smooth metric tensor \hat{g}_D . Let us consider another domain \tilde{D} . If \tilde{D} is another domain and $\psi : \tilde{D} \rightarrow D$ is a conformal map, we wish to reparametrize the metric on D by \tilde{D} via the change of coordinates $\psi(w) = x$ for $w \in \tilde{D}$ and $x \in D$. The transformation rule tells us that the metric on \tilde{D} , when parametrized by D using the map ψ^{-1} , takes the form

$$\hat{g}_{\tilde{D}} = \hat{g}_D(x) |\psi'(\psi^{-1}(x))|^{-2} dx^2 = \hat{g}_D(\psi(w)) |\psi'(w)|^{-2} dw^2. \quad (5.2)$$

The volume form is given by

$$\lambda_{\hat{g}_{\tilde{D}}}(dw) = \hat{g}_D(\psi(w)) dw$$

where dw is the Lebesgue measure on \tilde{D} . You can also decide to parametrize this metric space by D in which case the volume form is given by $\hat{g}_D(x) |(\psi^{-1})'(x)|^2 dx$. Note that we have the following relation

$$\int_{\tilde{D}} f(w) \lambda_{\hat{g}_{\tilde{D}}}(dw) = \int_D f(\psi^{-1}(x)) \hat{g}_D(x) |(\psi^{-1})'(x)|^2 \lambda(dx)$$

which ensures consistency, i.e. the integral of f and $f \circ \psi^{-1}$ are the same. The above relation says that $\hat{g}_D(x) |(\psi^{-1})'(x)|^2 \lambda(dx)$ is the image of $\lambda_{\hat{g}_{\tilde{D}}}(dw)$ by the map ψ . Recall that when ν is a measure on \tilde{D} then the image $\bar{\nu}$ of ν by ψ is the measure which satisfies

$$\int_D F(u) \bar{\nu}(du) = \int_{\tilde{D}} F(\psi(x)) \nu(dx).$$

Now we recall the conformal invariance property of the Green function. On the domain D , we consider the Green function G_D of the problem

$$\Delta_{\hat{g}_D} u = -2\pi f \text{ on } D, \quad \lim_{x \rightarrow \partial D} u(x) = 0.$$

If $\psi : \tilde{D} \rightarrow D$ is a conformal map and if we equip \tilde{D} with the metric $\hat{g}_{\tilde{D}}$ then the Green function $G_{\tilde{D}}$ of the problem

$$\Delta_{\hat{g}_{\tilde{D}}} u = -2\pi f \text{ on } \tilde{D}, \quad \lim_{x \rightarrow \partial \tilde{D}} u(x) = 0.$$

satisfies the relation

$$G_{\tilde{D}}(x, y) = G_D(\psi(x), \psi(y)). \quad (5.3)$$

Furthermore, the Green function on D does not depend on the smooth function g_D . In particular, if \tilde{D} is any domain conformally equivalent to the unit disk \mathbb{U} via a map $\psi : \tilde{D} \rightarrow \mathbb{U}$ then

$$G_{\tilde{D}}(x, y) = G_{\mathbb{U}}(\psi(x), \psi(y)) \quad (5.4)$$

where $G_{\mathbb{U}}$ is the Green function of the problem

$$\Delta u = -2\pi f \text{ on } \mathbb{U}, \quad \lim_{x \rightarrow \partial \mathbb{U}} u(x) = 0.$$

5.2 Free Fields in curved spaces

Since the Green function does not depend on the scalar metric tensor \hat{g}_D on the domain D , the law of the centered GFF $X_{\hat{g}_D}$ on D in the metric \hat{g}_D is that of a standard centered GFF X_D on D (i.e. in the flat metric). Because of (5.4), we have for all domain \tilde{D} conformally equivalent to D via $\psi : \tilde{D} \rightarrow D$ the following equality in law in the sense of (Schwartz) distributions

$$(X_{\hat{g}_{\tilde{D}}}(f))_{f \in C_c^\infty(\tilde{D})} \stackrel{\text{law}}{=} (X_D(f \circ \psi^{-1}))_{f \in C_c^\infty(\tilde{D})}. \quad (5.5)$$

Therefore, we can define simultaneously the GFF on every domain \tilde{D} conformally equivalent to \mathbb{U} by setting

$$(X_{\hat{g}_{\tilde{D}}}(f))_{f \in C_c^\infty(\tilde{D})} \stackrel{\text{law}}{=} (X_{\mathbb{U}}(f \circ \psi^{-1}))_{f \in C_c^\infty(\tilde{D})} \quad (5.6)$$

for a given GFF on \mathbb{U} with Dirichlet boundary condition.

Now we focus on the Free Field action in curved space, i.e. formally

$$S_D = \frac{1}{4\pi} \int_D [\langle \partial^{\hat{g}_D} \varphi, \partial^{\hat{g}_D} \varphi \rangle + Q R_{\hat{g}_D} \varphi] d\lambda_{\hat{g}_D} \quad (5.7)$$

because this turns the field φ into a shifted Gaussian Free Field. By plugging the exact value $R_{\hat{g}_D} = -\Delta_{\hat{g}_D} \ln \hat{g}_D$ of the curvature in this action and by performing a formal integration by parts, one can see that the field φ has the law of a GFF on D with mean given by $-\frac{Q}{2} \ln \hat{g}_D$. So we consider a centered GFF X on D with Dirichlet boundary condition and we set $\varphi_D = X - \frac{Q}{2} \ln \hat{g}_D$. Observe that our notation may appear misleading at first sight. The index D seems to indicate that the law of the field φ_D only depends on the domain D . Actually, it depends on the couple (D, \hat{g}_D) . Keep in mind this dependence in the following.

Let \tilde{D} be another domain and $\psi : \tilde{D} \rightarrow D$ is a conformal map. By using the relation between scalar metric tensors $\hat{g}_{\tilde{D}} = \hat{g}_D \circ \psi |\psi'|^{-2}$, we have the relation in law

$$\begin{aligned} \varphi_{\tilde{D}} &= X \circ \psi - \frac{Q}{2} \ln \hat{g}_{\tilde{D}} = X \circ \psi - \frac{Q}{2} \ln(\hat{g}_D \circ \psi |\psi'|^{-2}) \\ &= X \circ \psi - \frac{Q}{2} \ln \hat{g}_D \circ \psi + Q \ln |\psi'|. \end{aligned}$$

Therefore, we get the following reparametrization rule

$$\varphi_{\tilde{D}} \stackrel{\text{law}}{=} \varphi_D \circ \psi + Q \ln |\psi'| \quad (5.8)$$

when switching from a domain D to another domain \tilde{D} . This relation is standard in the physics literature as well as fundamental in the following.

5.3 Interaction term

The interaction term in the Liouville action is the term $e^{\gamma\varphi} \lambda_{\hat{g}}(dx)$ in (1.2). We will see that under the conditions $Q = \frac{\gamma}{2} + \frac{2}{\gamma}$ with $\gamma \in]0, 2]$, this term is conformally invariant under the reparametrization rule (5.8) and actually, it turns out that these are the only possible values to get a conformally invariant theory when restricted to the case $\gamma \in]0, 2]$.

In what follows, we will restrict to the case when $(D, \hat{g}_D) = (\mathbb{U}, \hat{g}_{\mathbb{U}})$ or any conformal reparametrization of such a metric space. The field $\varphi_{\mathbb{U}}$ will stand for the free field in the curved space $(\mathbb{U}, \hat{g}_{\mathbb{U}})$ as described in subsection 5.2. In what follows, the random measure $e^{\gamma\varphi_{\mathbb{U}}(x)} \lambda_{\hat{g}_{\mathbb{U}}}(dx)$

is constructed as explained in subsection 2.3 as the almost sure limit in the sense of weak convergence of measures

$$e^{\gamma\varphi_{\mathbb{U}}(x)}\lambda_{\hat{g}_{\mathbb{U}}}(\mathrm{d}x) = \lim_{\varepsilon \rightarrow 0} \varepsilon^{\frac{\gamma^2}{2}} e^{\gamma\varphi_{\varepsilon,\mathbb{U}}(x)}\lambda_{\hat{g}_{\mathbb{U}}}(\mathrm{d}x) \quad (5.9)$$

where $\varphi_{\varepsilon,\mathbb{U}}(x) = X_{\varepsilon}(x) - \frac{Q}{2} \ln \hat{g}_{\mathbb{U}}$ and $(X_{\varepsilon})_{\varepsilon}$ is any of the cut-off family of the free field X on \mathbb{U} discussed in subsection 2.3. It is plain to see that

$$e^{\gamma\varphi_{\mathbb{U}}(x)}\lambda_{\hat{g}_{\mathbb{U}}}(\mathrm{d}x) = \hat{g}_{\mathbb{U}}(x)^{-\frac{\gamma^2}{4}} C(x, \mathbb{U})^{\frac{\gamma^2}{2}} : e^{\gamma X(x)} : \mathrm{d}x. \quad (5.10)$$

Recall that, when $\gamma = 2$, a further $\sqrt{\ln \frac{1}{\varepsilon}}$ renormalizing term is needed in the above limit (5.9) (see [24, 25]).

For any conformal map $\psi : D \rightarrow \mathbb{U}$, we define

$$\varphi_{D,\varepsilon} = X_{\varepsilon} \circ \psi - \frac{Q}{2} \ln \hat{g}_D.$$

The interaction term $e^{\gamma\varphi_D(x)}\lambda_{\hat{g}_D}(\mathrm{d}x)$ on D is then define similarly to (5.9).

Remark 5.1. *We stress that the law of the limiting measure together with the field φ does not depend on the chosen cut-off approximation among (WN) or (CA) or (OBE). Actually this uniqueness in law holds at a more general level (see [45]).*

It may be worth stressing here that Gaussian multiplicative chaos theory allows us to define the measure $e^{\gamma\varphi_{\mathbb{U}}(x)}\lambda_{\hat{g}_{\mathbb{U}}}(\mathrm{d}x)$ on \mathbb{U} but it is not clear whether this measure assigns a finite mass to the whole set \mathbb{U} , especially because $\lambda_{\hat{g}_{\mathbb{U}}}(\mathbb{U}) = +\infty$. Yet we claim

Proposition 5.2. *Almost surely, we have*

$$\int_{\mathbb{U}} e^{\gamma\varphi_{\mathbb{U}}(x)}\lambda_{\hat{g}_{\mathbb{U}}}(\mathrm{d}x) < +\infty.$$

Proof. For $\gamma < 2$, the chaos representation (5.10) entails that

$$\mathbb{E} \left[\int_{\mathbb{U}} e^{\gamma\varphi_{\mathbb{U}}(x)}\lambda_{\hat{g}_{\mathbb{U}}}(\mathrm{d}x) \right] = \int_{\mathbb{U}} \hat{g}_{\mathbb{U}}(x)^{-\frac{\gamma^2}{4}} C(x, \mathbb{U})^{\frac{\gamma^2}{2}} \lambda_{\hat{g}_{\mathbb{U}}}(\mathrm{d}x).$$

Therefore, for $\gamma \in]0, 2[$, the mass of the unit disk is almost surely finite as it is plain to check that the latter quantity is finite. For $\gamma = 2$, the expectation of the measure $e^{2X(x)-2\mathbb{E}[X(x)^2]} \mathrm{d}x$ is not finite (see [24, 25]) so that the above argument does not directly applies. Yet, it is possible to use this argument to the truncated measure at level β , call it $e^{\gamma\varphi_{\mathbb{U},\beta}(x)}\lambda_{\hat{g}_{\mathbb{U}}}(\mathrm{d}x)$, as considered in [24]. Since this measure coincides with $e^{\gamma\varphi_{\mathbb{U}}(x)}\lambda_{\hat{g}_{\mathbb{U}}}(\mathrm{d}x)$ for β (random) large enough, the statement follows for $\gamma = 2$. As we do not want to introduce here all the background of cutlines at level β for the only purpose of proving Proposition 5.2, details are left to the reader. \square

Now we focus on the conformal invariance properties of the interaction term. The natural idea to construct the Liouville action (including the interaction term) is then to tilt the free field measure with the term $e^{-4\pi\mu \int_D e^{\gamma\varphi_D(x)}\lambda_{\hat{g}_D}(\mathrm{d}x)}$. However, Liouville field theory is a conformal field theory and, regarding to this point, it is important to check first that the measure $e^{\gamma\varphi_D(x)}\lambda_{\hat{g}_D}(\mathrm{d}x)$ is conformally invariant. Actually, this point is crucial because requiring this measure to be conformally invariant is equivalent to fixing the value of Q in terms of γ .

Roughly speaking, the measure $e^{\gamma\varphi_D(x)}\lambda_{\hat{g}_D}(dx)$ is expected to stand for the volume form of some "quantum metric" parameterized by (D, \hat{g}_D) (here the hyperbolic metric). To be intrinsic, the law of this quantum metric must be insensitive to the choice of any conformally equivalent reparametrization of (D, \hat{g}_D) . A conformal reparametrization of D is nothing but a pullback metric $\hat{g}_{\tilde{D}}$ on another domain \tilde{D} via a conformal $\psi : \tilde{D} \rightarrow D$. This pullback metric induces local deformations of the curvature and therefore another structure for the associated free field on curved space. The change of the law of the free field is exactly quantified by the reparametrization rule (5.8). At least heuristically, by writing x for the local coordinates on D and ψ for the local coordinates on \tilde{D} , we have

$$\begin{aligned} e^{\gamma\varphi_D} dx &= :e^{\gamma\varphi_D \circ \psi} : C(\psi, D)^{\frac{\gamma}{2}} |\psi'|^2 d\psi \\ &= :e^{\gamma\varphi_D \circ \psi} : C(\psi, \tilde{D})^{\frac{\gamma}{2}} |\psi'|^{2+\frac{\gamma}{2}} d\psi \\ &= e^{\gamma\varphi_{\tilde{D}}} d\psi \end{aligned}$$

provided that $\varphi_{\tilde{D}}(\psi) = \varphi_D(x) - Q \ln |\psi'|$, which is exactly the content of the parametrization rule (5.8). There is no difficulty in making a rigorous statement out of this and we claim

Proposition 5.3. *The measure $e^{\gamma\varphi_{\mathbb{U}}(x)}\lambda_{\hat{g}_{\mathbb{U}}}(dx)$ is conformally invariant. More precisely, given a conformal map $\psi : D \rightarrow \mathbb{U}$, we consider the random measure on D*

$$e^{\gamma\varphi_D(w)}\lambda_{\hat{g}_D}(dx).$$

This measure has the same law as the push forward of the measure $e^{\gamma\varphi_{\mathbb{U}}(x)}\lambda_{\hat{g}_{\mathbb{U}}}(dx)$ to D along ψ . Furthermore, if we construct the field φ_D with the help of the centered GFF $X \circ \psi$ on D then equality holds not only in law but also almost surely.

5.4 The Liouville field

We consider a cosmological constant $\mu \geq 0$ and a Liouville conformal factor $\gamma \in [0, 2]$. We set

$$Q = \frac{2}{\gamma} + \frac{\gamma}{2}.$$

Definition 5.4. *We define the law $\mathbb{P}_{\mu, \gamma, \hat{g}_D}$ of the Liouville field φ_D on (D, \hat{g}_D) associated to (μ, γ, \hat{g}_D) as*

$$\mathbb{E}_{\mu, \gamma, \hat{g}_D}[F(\varphi)] = Z_{\mu, \gamma, \hat{g}_D}^{-1} \mathbb{E}\left[F(\varphi_D) \exp\left(-4\pi\mu \int_D e^{\gamma\varphi_D(x)} \lambda_{\hat{g}_D}(dx)\right)\right]$$

where \mathbb{E} stands for the expectation with respect to the free field X , $\psi : D \rightarrow \mathbb{U}$ is a conformal map and

$$\varphi_D(x) = X \circ \psi(x) - \frac{Q}{2} \ln \hat{g}_D(x), \quad Z_{\mu, \gamma, \hat{g}_D} = \mathbb{E}\left[\exp\left(-4\pi\mu \int_D e^{\gamma\varphi_D(x)} \lambda_{\hat{g}_D}(dx)\right)\right]$$

and F is any bounded continuous functional on $H^{-1}(D)$.

Observe that this definition perfectly makes sense as we have seen that the integral $\int_D e^{\gamma\varphi_D(x)} \lambda_{\hat{g}_D}(dx)$ appearing in the above expectations is almost surely finite. Furthermore, we stress that the law of this field corresponds to the Liouville action (1.2). Also, as a consequence of Proposition 5.3, we claim

Proposition 5.5. (Conformal invariance) *For any conformal map $\psi : \tilde{D} \rightarrow D$, the law under $\mathbb{P}_{\mu, \gamma, \hat{g}_{\tilde{D}}}$ of the random distribution $\varphi_D \circ \psi + Q \ln |\psi'(x)|$ on \tilde{D} is the same as the law of $\varphi_{\tilde{D}}$ under $\mathbb{P}_{\mu, \gamma, \hat{g}_{\tilde{D}}}$.*

5.5 Physical metrics

One of the main issue in Liouville theory is the study of metrics on \mathbb{U} defined for $\alpha \in]0, 2]$ by

$$g(x) = e^{\alpha\varphi_{\mathbb{U}}(x)} g_{\mathbb{U}}(x) dx^2$$

where $\varphi_{\mathbb{U}}$ is the Liouville field with parameters $\mu \geq 0$, $\gamma \in]0, 2]$ and dx^2 stands for the Euclidean metric on \mathbb{U} . Absolute continuity of the law of the Liouville field with respect to the law of the GFF on \mathbb{U} with Dirichlet boundary conditions is convenient to extend properties that are true almost surely with respect to the Free Field law (when you switch off the interaction term, i.e. $\mu = 0$) to the general case of non critical LFT (when you switch on the interaction term, i.e. $\mu > 0$). This is the case of the existence of the metric, which we explain below. By metric, we mean volume form, Brownian motion, semi-group, Laplace-Beltrami operator and existence of a heat kernel associated to g . It just suffices to tilt the law of the object constructed in the Free Field context ($\mu = 0$) by the term $\exp\left(-4\pi\mu \int_{\mathbb{U}} e^{\gamma\varphi_{\mathbb{U}}(x)} \lambda_{\hat{g}_{\mathbb{U}}}(dx)\right)$.

We first illustrate our claims with the construction of the volume form. In the case $\mu = 0$ and $\alpha \in]0, 2[$, we know from subsection 2.3 that the family

$$\left(\varepsilon^{\frac{\alpha^2}{2}} e^{\alpha\varphi_{\mathbb{U},\varepsilon}(x)} \lambda_{\hat{g}_{\mathbb{U}}}(dx)\right)_{\varepsilon} \quad \text{with} \quad \varphi_{\mathbb{U},\varepsilon}(x) = X_{\varepsilon}(x) - \frac{Q}{2} \ln \hat{g}_{\mathbb{U}}(x)$$

almost surely weakly converges towards a random measure, which we call $e^{\alpha\varphi_{\mathbb{U}}(x)} \lambda_{\hat{g}_{\mathbb{U}}}(dx)$ on \mathbb{U} . By absolute continuity of the law of the Liouville field with respect to the GFF law, we deduce that, under $\mathbb{P}_{\mu,\gamma,\hat{g}_{\mathbb{U}}}$, the family $\left(\varepsilon^{\frac{\alpha^2}{2}} e^{\alpha\varphi_{\varepsilon,\mathbb{U}}(x)} \lambda_{\hat{g}_{\mathbb{U}}}(dx)\right)_{\varepsilon}$ almost surely weakly converges towards the same random measure $e^{\alpha\varphi_{\mathbb{U}}(x)} \lambda_{\hat{g}_{\mathbb{U}}}(dx)$. However, observe that, for $\mu > 0$, the law of $e^{\alpha\varphi_{\mathbb{U}}(x)} \lambda_{\hat{g}_{\mathbb{U}}}(dx)$ under \mathbb{P} (the law of the GFF X) differs from its law under the probability measure $\mathbb{P}_{\mu,\gamma,\hat{g}_{\mathbb{U}}}$. Also, the measure $e^{\alpha\varphi_{\mathbb{U}}(x)} \lambda_{\hat{g}_{\mathbb{U}}}(dx)$ is atomless under $\mathbb{P}_{\mu,\gamma,\hat{g}_{\mathbb{U}}}$ as it is under \mathbb{P} .

Concerning the Liouville Brownian motion, the construction of [30, 31] can be adapted as follows. We consider a standard planar Brownian motion B on \mathbb{R}^2 . For $x \in \mathbb{U}$, we set $B_t^x = x + B_t$ and

$$\tau_x^{\mathbb{U}} = \inf\{s > 0, B_s^x \notin \mathbb{U}\}.$$

Then we define the increasing additive functional $F^{\mathbb{U},\alpha}(x, t)$ as the almost sure limit

$$\forall t \geq 0, \quad F^{\mathbb{U},\alpha}(x, t) = \lim_{\varepsilon \rightarrow 0} \int_0^{t \wedge \tau_x^{\mathbb{U}}} e^{\alpha\varphi_{\mathbb{U},\varepsilon}(B_r^x)} dr. \quad (5.11)$$

Similarly to the volume form, one can check that the limit is equal to

$$\forall t \geq 0, \quad F^{\mathbb{U},\alpha}(x, t) = \int_{\mathbb{R}^2} C(z, \mathbb{U})^{\frac{\alpha^2}{2}} g_{\mathbb{U}}(z)^{1-\frac{Q\alpha}{2}} e^{\alpha X(z) - \frac{\alpha^2}{2} \mathbb{E}[X^2(z)]} \nu_{t \wedge \tau_x^{\mathbb{U}}}(dz), \quad (5.12)$$

where $\nu_{t \wedge \tau_x^{\mathbb{U}}}$ is the occupation measure of the Brownian motion B^x up to time $t \wedge \tau_x^{\mathbb{U}}$. This PCAF can be extended to the whole of \mathbb{U} (see [30]), i.e. can be defined almost surely for all starting point $x \in \mathbb{U}$. This extension, still denoted by $F^{\mathbb{U},\alpha}(x, t)$, is continuous and strictly increasing up to time $t < \tau_x^{\mathbb{U}}$. We denote by $F^{\mathbb{U},\alpha}(x, t)^{-1}$ the reciprocal function and we set $\tilde{\tau}_x^{\mathbb{U}} = \lim_{t \rightarrow \tau_x^{\mathbb{U}}} F^{\mathbb{U},\alpha}(x, t)$. The Liouville Brownian motion \mathcal{B} is then defined as

$$\forall x \in \mathbb{U}, \forall t < \tilde{\tau}_x^{\mathbb{U}}, \quad \mathcal{B}_t^{\alpha,x} = B_{F^{\mathbb{U},\alpha}(x,t)^{-1}}^x. \quad (5.13)$$

This construction is carried out in [30] for $\mu = 0$. For $\mu > 0$, the law of the Liouville Brownian motion is nothing but the law of \mathcal{B}^α under the probability measure $\mathbb{P}_{\mu, \gamma, \hat{g}_\mathbb{U}}$. By adapting the argument of the volume form, it is plain to see that the Liouville Brownian motion \mathcal{B}^γ is conformally invariant under the reparametrization rule (5.8).

One can also follow [30, 31] to construct the associated semi-group, Laplace-Beltrami operator, heat kernel, resolvent and Dirichlet form by using the argument of absolute continuity. We can also claim that the spectral dimension of the metric $e^{\alpha\varphi}$ is 2 for $\alpha \in]0, 2[$, $\gamma \in]0, 2[$ and $\mu \geq 0$ with the help of [44]. Let us just stress that the case $\alpha = 2$ is technically more subtle and one must state properly the properties of the metric $e^{\alpha\varphi}$ as explained in [47].

5.6 Semiclassical limit and large deviations

Let us just stress that all the results established in section 3 excepted that you replace the classical Liouville equation in flat space (3.3) by the she equation in curved space

$$\Delta_{\hat{g}_\mathbb{U}} U - R_{\hat{g}_\mathbb{U}} = 8\pi^2 \Lambda e^U, \quad U|_{\partial\mathbb{U}} = 0. \quad (5.14)$$

The same remark holds for section 4 where you replace (4.3) by

$$\Delta_{\hat{g}_\mathbb{U}} U - R_{\hat{g}_\mathbb{U}} = 8\pi^2 \Lambda e^U - 2\pi \sum_{i=1}^p \chi \delta_{z_i}, \quad U|_{\partial\mathbb{U}} = 0. \quad (5.15)$$

5.7 Conformal weights

Here we study how the measure $e^{\alpha\varphi_\mathbb{U}(x)} \lambda_{\hat{g}_\mathbb{U}}(dx)$ changes under conformal reparametrization. To quantify this change, we will introduce the notion of conformal weight. Recall that the Liouville field can be simultaneously defined on all the domains D conformally equivalent to \mathbb{U} via a conformal map ψ by

$$\varphi_{\tilde{D}}(w) = X(\psi(w)) - \frac{Q}{2} \ln(\hat{g}_\mathbb{U} \circ \psi(w) |\psi'(w)|^{-2}).$$

Definition 5.6. (Conformal weight) *The conformal weight of the (spinless) operator $e^{\alpha\varphi_\mathbb{U}}$ is defined as the exponent Δ_α such that*

$$\int_{\psi(A)} e^{\alpha\varphi_\mathbb{U}(x)} \lambda_{\hat{g}_\mathbb{U}}(dx) = \int_A |\psi'(w)|^{2-2\Delta_\alpha} e^{\alpha\varphi_D(w)} \lambda_{\hat{g}_D}(dw).$$

Proposition 5.7. *The conformal weight of the operator $e^{\alpha\varphi_D(x)} \lambda_{\hat{g}_D}(dx)$ is given by*

$$\Delta_\alpha = \frac{\alpha Q}{2} - \frac{\alpha^2}{4}.$$

This is a rather elementary computation in the same spirit as the proof of Proposition 5.3. The proof is thus left as an exercise. We stress that conformal invariance is equivalent to having conformal weight 1. We thus recover the fact that the operator $e^{\alpha\varphi_\mathbb{U}}$ is conformally invariant only for $\gamma = \alpha$.

5.8 KPZ formulae

Geometrical KPZ formula

In this section, we explain the KPZ formula [37, 12, 19], at least as formulated in [46] (or [4, 25]; see also [23] for another formulation). The geometrical KPZ formula is a relation between the Hausdorff dimensions of a given set $A \subset \mathbb{U}$ as measured by the Lebesgue measure or the random measure $e^{\alpha\varphi_{\mathbb{U}}(x)} \lambda_{\hat{g}_{\mathbb{U}}}(\mathrm{d}x)$. Recall that, given an atomless Radon measure λ on D and $s \in [0, 1]$, we define

$$H_{\lambda}^{s,\delta}(A) = \inf \left\{ \sum_k \lambda(B_k)^s \right\}$$

where the infimum runs over all the coverings $(B_k)_k$ of A with closed Euclidean balls (non necessarily centered at A) with radius $r_k \leq \delta$. We define the s -dimensional λ -Hausdorff measure:

$$H_{\lambda}^s(A) = \lim_{\delta \rightarrow 0} H_{\lambda}^{s,\delta}(A).$$

The limit exists but may be infinite and defines a metric outer measure on the σ -field of H_{μ}^s -measurable subsets of D , which contains all the Borel sets. The λ -Hausdorff dimension of the set A is then defined as the value

$$\dim_{\lambda}(A) = \inf\{s \geq 0; H_{\lambda}^s(A) = 0\} = \sup\{s \geq 0; H_{\lambda}^s(A) = +\infty\}. \quad (5.16)$$

Notice that $\dim_{\lambda}(A) \in [0, 1]$ and a (non standard) Frostman lemma can be proved to characterize the λ -Hausdorff dimension.

In what follows, given a compact set K of D , we define its Hausdorff dimensions $\dim_{\lambda_{\hat{g}_D}}(K)$ and $\dim_{\alpha}(K)$ computed as indicated above with λ respectively equal to the volume form $\lambda_{\hat{g}_D}(\mathrm{d}x)$ or the random measure $e^{\alpha\varphi_D} \lambda_{\hat{g}_D}$, both of which are atom free. We claim

Theorem 5.8. (KPZ formula). *Let K be a compact set of D and $\gamma \in]0, 2]$, $\mu \geq 0$. $\mathbb{P}_{\mu,\gamma,\hat{g}_D}$ -almost surely, we have the relation*

$$\dim_{\lambda_{\hat{g}_D}}(K) = \left(1 + \frac{\alpha^2}{4}\right) \dim_{\alpha}(K) - \frac{\alpha^2}{4} \dim_{\alpha}(K). \quad (5.17)$$

Proof. Let K be a compact set of D with $\lambda_{\hat{g}_D}$ -Hausdorff dimension $\dim_{\lambda_{\hat{g}_D}}(K)$. Since the \hat{g}_D -metric is locally isometric to the Euclidean metric on D , the set K also has Euclidean λ_D -Hausdorff dimension $\dim_{\lambda_{\hat{g}_D}}(K)$. From [46], we deduce that the set K has Hausdorff dimension $\dim_{\alpha}(K)$ that is related to $\dim_{\lambda_{\hat{g}_D}}(K)$ via the relation (5.8) \mathbb{P} -almost surely (i.e. for $\mu=0$). Therefore this is also true $\mathbb{P}_{\mu,\gamma,\hat{g}_D}$ -almost surely by absolute continuity. \square

KPZ scaling law

Here we focus on the KPZ scaling law [37]. We want to establish the following scaling relation

Theorem 5.9. *We have*

$$\mathbb{E}_{\mu,\gamma,\hat{g}_{\mathbb{U}}} \left[\int_{A_1} e^{\alpha_1 \varphi_{\mathbb{U}}} \lambda_{\hat{g}_{\mathbb{U}}} \dots \int_{A_n} e^{\alpha_n \varphi_{\mathbb{U}}} \lambda_{\hat{g}_{\mathbb{U}}} \right] = C \mu^{\frac{4}{\gamma^2} \left(n - \frac{\gamma}{2} \sum_{i=1}^n \alpha_i \right)} \quad (5.18)$$

for every possible disjoint sets $A_1, \dots, A_n \subset \mathbb{U}$, $\alpha_1, \dots, \alpha_n \in [0, 2]$ and

$$C = \mathbb{E}_{1,\gamma,g=\mu^{-\frac{4}{\gamma^2}} \hat{g}_{\mathbb{U}}} \left[\int_{A_1} e^{\alpha_1 \varphi_g} \lambda_g(\mathrm{d}x) \dots \int_{A_n} e^{\alpha_n \varphi_g} \lambda_g(\mathrm{d}x) \right].$$

Remark 5.10. Notice here that the KPZ scaling law is not that claimed by [37]: the power of μ computed in [37] is rather $\mu^{\frac{2}{\gamma}(Q-\sum_{i=1}^n \alpha_i)}$. There are several reasons. The first one is that Liouville field theory is here not formulated on the sphere, which is structurally far different from the hyperbolic disk. The formula in [37] results from the Gauss-Bonnet theorem and it can only be applied on compact surfaces: this is not the case of the hyperbolic disk. Furthermore [37] deals with Liouville quantum gravity, meaning that the authors also average over background metrics: this allows to absorb the background metric dependence in the constant C . Finally, by considering probability laws, we are actually working with a renormalized version of physicists' partition function. Physicist rather work with

$$\mathbb{E} \left[\int_{A_1} e^{\alpha_1 \varphi_{\mathbb{U}}} \lambda_{\hat{g}_{\mathbb{U}}} \dots \int_{A_n} e^{\alpha_n \varphi_{\mathbb{U}}} \lambda_{\hat{g}_{\mathbb{U}}} e^{\int_{\mathbb{U}} \varphi_{\mathbb{U}} R_{\hat{g}_{\mathbb{U}}} \lambda_{\hat{g}_{\mathbb{U}}} - 4\pi \mu \int_{\mathbb{U}} e^{\gamma \varphi_{\mathbb{U}}} \lambda_{\hat{g}_{\mathbb{U}}}} \right]$$

(where \mathbb{E} stands for expectation with respect to the centered GFF together with averaging over metrics), which should scale on the sphere like $\mu^{\frac{2}{\gamma}(Q-\sum_{i=1}^n \alpha_i)}$ whereas we are actually working with this the renormalized partition function.

Proof. Consider $L > 0$. Let us denote by $\varphi_{\hat{g}}$ the Liouville field associated to the metric $\hat{g} = L\hat{g}_{\mathbb{U}}$ on \mathbb{U} . Observe that

$$\varphi_{\hat{g}} = \varphi_{\mathbb{U}} - \frac{Q}{2} \ln L.$$

Therefore

$$\begin{aligned} \mathbb{E}_{1, \gamma, \hat{g} = \mu^{-\frac{4}{\gamma^2}} \hat{g}_{\mathbb{U}}} & \left[\int_{A_1} e^{\alpha_1 \varphi_{\hat{g}}} \lambda_{\hat{g}} \dots \int_{A_n} e^{\alpha_n \varphi_{\hat{g}}} \lambda_{\hat{g}} \right] \\ &= \mathbb{E} \left[\int_{A_1} e^{\alpha_1 (\varphi_{\mathbb{U}} - \frac{Q}{2} \ln L)} L \lambda_{\hat{g}_{\mathbb{U}}} \dots \int_{A_n} e^{\alpha_n (\varphi_{\mathbb{U}} - \frac{Q}{2} \ln L)} L \lambda_{\hat{g}_{\mathbb{U}}} e^{-4\pi \int_{\mathbb{U}} e^{\gamma (\varphi_{\mathbb{U}} - \frac{Q}{2} \ln L)} L \lambda_{\hat{g}_{\mathbb{U}}}} \right] \\ &= L^{n - \frac{Q}{2} \sum_i \alpha_i} \mathbb{E} \left[\int_{A_1} e^{\alpha_1 \varphi_{\mathbb{U}}} \lambda_{\hat{g}_{\mathbb{U}}} \dots \int_{A_n} e^{\alpha_n \varphi_{\mathbb{U}}} \lambda_{\hat{g}_{\mathbb{U}}} e^{-4\pi L^{1-\gamma Q/2} \int_{\mathbb{U}} e^{\gamma \varphi_{\mathbb{U}}} \lambda_{\hat{g}_{\mathbb{U}}}} \right]. \end{aligned}$$

The result follows by setting $L = \mu^{-\frac{4}{\gamma^2}}$, that is $\mu = L^{1-\frac{\gamma Q}{2}}$. \square

A Solving the modified Liouville equation

This section is devoted to solving the (eventually singular) Liouville equation as well as some variants. The technics developed here are known in the community of differential geometry and are close to [6]. Yet, we have not found references corresponding exactly to the results we need. Furthermore, the proofs are rather elementary and may help the reader (not necessarily familiar with these equations) to understand how it works.

Theorem A.1. *For every function f belonging to $H_0^1(\mathbb{U})$, the equation*

$$\Delta U = 8\pi^2 \Lambda e^U - 2\pi f, \quad U|_{\partial \mathbb{U}} = 0 \tag{A.1}$$

admits a weak solution on \mathbb{U} which is Hölder continuous $C^{1,\alpha}(\mathbb{U})$ for all $\alpha < 1$.

Proof. Let us consider the solution $g \in H_0^1(\mathbb{U})$ of the equation $\Delta g = -2\pi f$ with boundary condition $g|_{\partial \mathbb{U}} = 0$. Let us set $h(x) = 8\pi^2 \Lambda e^{g(x)}$. It is then readily seen that U is a weak solution to (A.1) if and only if $V = U - g$ is a weak solution to

$$\Delta V = h(x) e^{V(x)}, \quad V|_{\partial \mathbb{U}} = 0. \tag{A.2}$$

With the help of Sobolev-Orlicz space embeddings [55], $H_0^1(\mathbb{U})$ is continuously embedded into the Orlicz space with Young function $\Phi(t) = \exp(t^2) - 1$. It results that $e^g \in \mathbb{L}^p(\mathbb{U})$ for all $p > 1$.

Let us consider the positive functional E defined on $H_0^1(\mathbb{U})$

$$E(V) = \int_{\mathbb{U}} (|\partial V(x)|^2 + 2h(x)e^{V(x)}) dx := \int_{\mathbb{U}} F(x, V(x), \partial V(x)) dx.$$

Since $h \in \mathbb{L}^q(\mathbb{U})$ for some $q > 1$, the functional E is indeed defined on $H_0^1(\mathbb{U})$. Since $p \mapsto F(x, V, p)$ is convex and F is greater or equal to 0 the functional E is weakly lower semi-continuous (see [52, Theorem 1.6]). Since $E(V)$ goes to infinity as $\int_{\mathbb{U}} |\partial V|^2 dx$ goes to infinity, $E(V)$ achieves its infimum in $H_0^1(\mathbb{U})$ as a consequence of [52, Theorem 1.2]. One can check that $\operatorname{argmin} E$ is reduced to one point (E is strictly convex) which is a weak solution to (A.1): we call it V . Once again, with the help of Sobolev-Orlicz space embeddings, we know that $e^V \in \mathbb{L}^p(\mathbb{U})$. By Hölder's inequality, the product $he^V = \Delta V$ belongs to $\mathbb{L}^p(\mathbb{U})$ for all $p > 1$. Hence for any $p > 2$ and $q < 2$ which are Hölder conjugates

$$\begin{aligned} |V(x) - V(y)| &= 4\pi\Lambda \left| \int_{\mathbb{U}} e^{V(z)+g(z)} (G_{\mathbb{U}}(x, z) + G_{\mathbb{U}}(y, z)) dz \right| \\ &\leq \left(\int_{\mathbb{U}} e^{p(V(z)+g(z))} dz \right)^{1/p} \left(\int_{\mathbb{U}} |G_{\mathbb{U}}(x, z) + G_{\mathbb{U}}(y, z)|^q dz \right)^{1/q} \\ &\leq C|x - y|^{2/q} |\ln |x - y||. \end{aligned}$$

This allows us to conclude that V is Hölder continuous on \mathbb{U} . \square

Proposition A.2. *For every function f, h belonging to $H_0^1(\mathbb{U})$, we denote by U_t the solution of the equation*

$$\Delta U_t = 8\pi^2 \Lambda e^{U_t} - 2\pi(f + th), \quad U_t|_{\partial\mathbb{U}} = 0. \quad (\text{A.3})$$

Then the family $(\frac{U_t - U_0}{t})_{t>0}$ strongly converges in $H_0^1(\mathbb{U})$ towards the solution V of the equation

$$\Delta V = 8\pi^2 \Lambda e^{U_0} - 2\pi h, \quad V|_{\partial\mathbb{U}} = 0. \quad (\text{A.4})$$

Proof. First notice that (A.4) is linear in V so that there are no troubles in establishing existence and uniqueness of a weak solution to this equation (see e.g. [52, Theorem 1.2]). Furthermore, the Sobolev-Orlicz embedding entails that $\sup_{t \in [0,1]} \int_{\mathbb{U}} e^{2U_t} dx < +\infty$ for all $p > 1$ and hence (from (A.3)) that ΔU is in $\mathbb{L}^2(\mathbb{U})$. The standard Sobolev embedding then entails that $M = \sup_{t \in [0,1]} \sup_{x \in \mathbb{U}} |U_t(x)| < +\infty$. In what follows, we will consider a constant D such that

$$|e^x - 1 - x| \leq Dx^2, \quad \text{for all } |x| \leq 2M. \quad (\text{A.5})$$

Set $V_t = \frac{U_t - U_0}{t}$. Furthermore, by considering the difference of (A.3) evaluated at t and $t = 0$ and then integrating against a test function ϕ in $H_0^1(\mathbb{U})$, we obtain

$$\int_{\mathbb{U}} \langle \partial V_t, \partial \phi \rangle dx + 8\pi^2 \Lambda \int_{\mathbb{U}} e^{U_0} t^{-1} (e^{U_t - U_0} - 1) \phi dx = 2\pi \int_{\mathbb{U}} h \phi dx. \quad (\text{A.6})$$

Taking $\phi = V_t$ and using the inequality $x(e^x - 1) \geq 0$, we deduce

$$\int_{\mathbb{U}} |\partial V_t|^2 dx \leq 2\pi \int_{\mathbb{U}} h V_t dx \leq C \left(\int_{\mathbb{U}} |\partial V_t|^2 dx \right)^{1/2} \left(\int_{\mathbb{U}} |h|^2 dx \right)^{1/2}. \quad (\text{A.7})$$

We used the Poincaré inequality to get the last inequality. Hence the sequence $(V_t)_t$ is bounded in $H_0^1(\mathbb{U})$, and has limit points when $t \rightarrow 0$ for the weak topology in $H_0^1(\mathbb{U})$. Let \bar{V} be one of these limit points. By taking the limit along a subsequence converging to \bar{V} in (A.6) (and using (A.5) to get rid of the exponential term), we deduce that \bar{V} is a weak solution to (A.4). By uniqueness, $\bar{V} = V$ and is the weak limit of $(V_t)_t$. It remains to prove the convergence of the norms to get the strong convergence. By taking once again $\phi = V_t$ in (A.6), we get

$$\lim_{t \rightarrow 0} \left(\int_{\mathbb{U}} |\partial V_t|^2 dx + 8\pi^2 \Lambda \int_{\mathbb{U}} e^{U_0} t^{-1} (e^{U_t - U_0} - 1) V_t dx \right) = 2\pi \int_{\mathbb{U}} hV dx.$$

The main difficult term is the integral containing the exponential term. With the help of (A.5), we have

$$\int_{\mathbb{U}} e^{U_0} t^{-1} (e^{U_t - U_0} - 1) V_t dx = \int_{\mathbb{U}} e^{U_0} |V_t|^2 dx + H_t, \quad |H_t| \leq Dt \int_{\mathbb{U}} |V_t|^3 dx.$$

By the Rellich-Kondrachov theorem, the embedding $H_0^1(\mathbb{U}) \rightarrow L^2(\mathbb{U})$ is compact so that the first term in the right-hand side converges towards $\int_{\mathbb{U}} e^{U_0} |V|^2 dx$. Furthermore as V_t is bounded in $H_0^1(\mathbb{U})$, the Sobolev embedding entails that $\sup_{t \in [0,1]} \int_{\mathbb{U}} e^{U_0} |V_t|^3 dx < +\infty$. Hence the second term goes to 0. We deduce

$$\lim_{t \rightarrow 0} \int_{\mathbb{U}} |\partial V_t|^2 dx = -8\pi^2 \Lambda \int_{\mathbb{U}} e^{U_0} V^2 dx + 2\pi \int_{\mathbb{U}} hV dx = \int_{\mathbb{U}} |\partial V|^2 dx.$$

The proof is complete. \square

Proposition A.3. *Assume that the family $(f_t)_{t>0}$ is weakly converging towards f_0 in $H_0^1(\mathbb{U})$ as $t \rightarrow 0$. Denote by U_t the solution of the equation*

$$\Delta U_t = 8\pi^2 \Lambda e^{U_t} - 2\pi f_t, \quad U_t|_{\partial \mathbb{U}} = 0. \quad (\text{A.8})$$

Then the family $(U_t)_{t>0}$ strongly converges in $H_0^1(\mathbb{U})$ towards U_0 .

Proof. The key points are first to observe that $(f_t)_{t>0}$ is strongly converging towards f_0 in $L^2(\mathbb{U})$ by using the Rellich-Kondrachov Theorem and that $\sup_{t>0} \int_{\mathbb{U}} |f_t|^p dx < +\infty$ by the Sobolev embeddings. Then the arguments are quite similar to the proof of Proposition A.2: we can prove that $U_t - U_0$ converges strongly to the a solution of (A.4) with $h = 0$. Details are thus left to the reader. \square

Theorem A.4. *Consider $z_1, \dots, z_p \in \mathbb{U}$ and $\chi_1, \dots, \chi_p \in]0, 2[$. The equation*

$$\Delta U = 8\pi^2 \Lambda e^U - 2\pi \sum_{i=1}^p \chi_i \delta_{z_i}, \quad U|_{\partial \mathbb{U}} = 0 \quad (\text{A.9})$$

admits a solution on \mathbb{U} such that $U - \sum_{i=1}^p \chi_i G_{\mathbb{U}}(\cdot, z_i)$ is locally Hölder continuous on \mathbb{U} .

Proof. By using the same trick as in the proof of Theorem A.1, by setting $V = U - \sum_{i=1}^p \chi_i G_{\mathbb{U}}(\cdot, z_i)$, it suffices to solve the equation

$$\Delta V(x) = h(x) e^{V(x)}, \quad V|_{\partial \mathbb{U}} = 0 \quad (\text{A.10})$$

with $h(x) = 8\pi^2 \Lambda e^{\sum_{i=1}^p \chi_i G_{\mathbb{U}}(\cdot, z_i)}$. Let us consider the functional

$$E(V) = \int_{\mathbb{U}} |\partial V(x)|^2 + 2h(x) e^{V(x)} dx$$

defined on $H_0^1(\mathbb{U})$ (observe that $h \in L^q(\mathbb{U})$ for some $q < q_\chi = 2/\max_{i \in \{1, \dots, p\}} \chi_i$ where $q_\chi > 1$). Therefore, one can use the same arguments than the ones in the proof of Theorem A.1 to deduce from [52, Theorem 1.2] and [52, Theorem 1.6] that the functional E is weakly lower semi-continuous and achieves its infimum in $H_0^1(\mathbb{U})$. Moreover the reader can check that $\argmin E$ is reduced to a point (by convexity) which is a solution of (A.10). Let us call it V . With the help of Sobolev-Orlicz space embeddings [55], $H_0^1(\mathbb{U})$ is continuously embedded into the Orlicz space with Young function $\Phi(t) = \exp(t^2) - 1$. It results that $e^V \in \mathbb{L}^p(\mathbb{U})$ for all $p > 1$. By Hölder's inequality, the product he^V belongs to $\mathbb{L}^q(\mathbb{U})$ for all $q < q_\chi$. Standard arguments of Sobolev embeddings allows us to conclude that V is α -Hölder continuous on \mathbb{U} for all $\alpha < \max(1, 2(1 - q_\chi^{-1}))$. \square

Proposition A.5. *For each function $f \in H_0^1(U)$ on \mathbb{U} , the equation*

$$\Delta U = 4\pi\Lambda e^U - 2\pi \sum_{i=1}^p \chi_i \delta_{z_i} + 2\pi f, \quad U|_{\partial\mathbb{U}} = 0 \quad (\text{A.11})$$

admits a solution on \mathbb{U} such that $U - \sum_{i=1}^p \chi_i G_{\mathbb{U}}(\cdot, z_i)$ is locally Hölder continuous on \mathbb{U} .

Proof. It suffices to adapt the arguments of Theorem A.4. \square

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